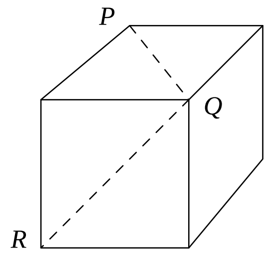
2012-2013 Middle Solutions Round 2

M1. PQ and QR are diagonals on two faces of a cube as shown. What is the size of $\angle PQR$?



Solution

All the face diagonals of a cube are equal so $PR = PQ = QR$. So PQR is an equilateral triangle, therefore $\angle PQR = 60°$.

M2. Very shortly after leaving Elmouth by bus on the half-hour ride to Elwick one day, we met a bus coming towards us. I wondered to myself how many such buses we should meet before we reached our destination. There is a ten minute service each way.

I assumed that all buses concerned travelled on time and at a constant speed. How many buses should we have met before we reached our destination?

Solution

As the buses depart at 10 minute intervals and travel towards each other, there will be 'passings' every 5 minutes.

Let the first passing be after y minutes.

It must be the case that $y < 5$ otherwise it would not be the first encounter.

So, buses will pass as follows: y, $y + 5$, $y + 10$, $y + 15$, $y + 20$, $y + 25$. These are all less than 30 but the next one would exceed 30.

So there will be 6 buses.

M3. Susan has 20 coins in her purse. She has only 10p, 20p and 50p coins and their total value is £5. She has more 50p coins than 10p coins. How many coins of each type does she have?

Solution

Let the number of 10p coins be x, 20p coins y and 50p coins z. Then

$$x + y + z = 20 \qquad \text{(20 coins)}$$

$$10x + 20y + 50z = 500 \qquad \text{(value 500p)}$$

$$\text{i.e.} \qquad x + 2y + 5z = 50$$

Eliminating y gives

$$3z - x = 10$$

x and z must both be non-negative so $z > 3$ and the first solution is

$$z = 4, x = 2$$

and others are

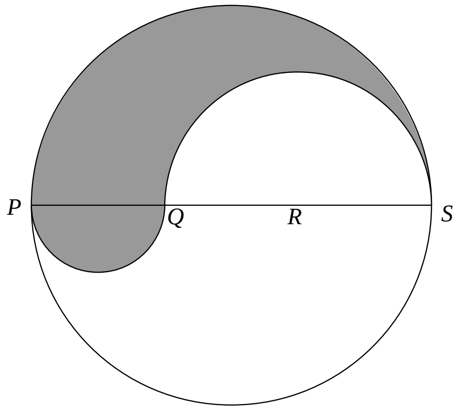
z	5	6	7	…
x	5	8	11	…

Only the first solution has $z > x$, i.e. more 50p coins than 10p coins.

In this case $y = 14$.

So she has two 10p coins, fourteen 20p coins and four 50p coins.

M4. $PQRS$ is the diameter of a circle whose radius is r. The lengths PQ, QR and RS are equal. Semicircles are drawn on PQ and QS to create the shaded figure shown in the diagram. Find the perimeter of the shaded figure.



Solution

Large semicircle, radius r, length of curve πr.

Medium semicircle, radius $\frac{2}{3}r$, length of curve $\frac{2}{3}\pi r$.

Small semicircle, radius $\frac{1}{3}r$, length of curve $\frac{1}{3}\pi r$.

Total perimeter of shaded area $\pi r + \frac{2}{3}\pi r + \frac{1}{3}\pi r = 2\pi r$.

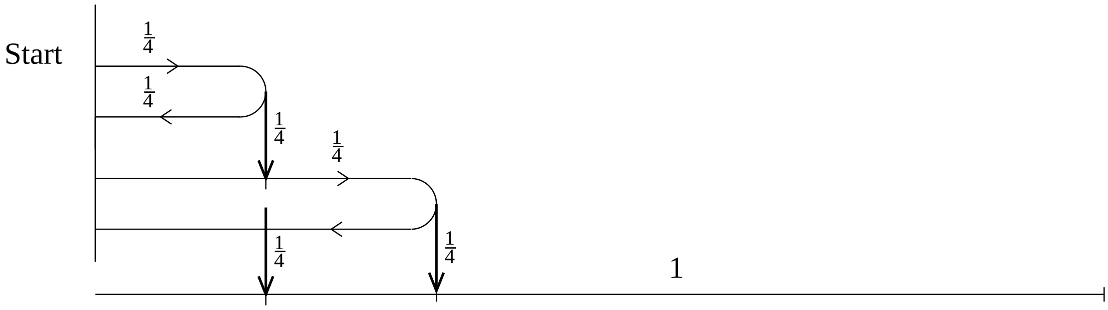
M5. Show that the maximum range of an aeroplane is extended by a factor of $\frac{1}{3}$ when there is a second identical support plane which sets off at the same time to provide in-air refuelling. The support plane must return safely to the starting point.

Now consider the situation where there are two identical support planes which can (instantaneously) refuel each other or the original plane as required. The support planes set off at the same time as the original plane and both must return safely to the starting point. By how much can the maximum range of the plane be extended?

Solution 1

Both planes fly $\frac{1}{3}$ of their maximum range, using $\frac{1}{3}$ of their fuel capacity, and then the support plane transfers $\frac{1}{3}$ of its fuel capacity to the first plane. The support plane uses the remaining $\frac{1}{3}$ of its fuel capacity to return safely to base and the first plane is full of fuel. Thus the range of the first plane has been extended by $\frac{1}{3}$.

All three planes fly $\frac{1}{4}$ of their maximum range, and then one support plane refuels each of the others with $\frac{1}{4}$ of its fuel capacity, leaving itself with $\frac{1}{4}$ to fly home and each of the other planes full to capacity. The two remaining planes fly another $\frac{1}{4}$ of their maximum range before the second support plane transfers $\frac{1}{4}$ of its fuel capacity to the long distance plane, leaving itself with $\frac{1}{2}$ of its fuel capacity with which to return safely and the long distance plane full of fuel. Thus the range of the first plane has been extended by $\frac{1}{2}$.



Solution 2

Let the original range of the plane be r.

With two planes, let the support plane fly x out and x back, and let the main plane fly x out and then a further y. The total distances flown by the two planes must be $2r$, so

$$3x + y = 2r.$$

To maximise $x + y$ subject to this constraint we must make y as large as possible. To do this, let the second plane be full after refuelling, so that $y = r$. This makes $x = \frac{1}{3}r$, so the extension is $x + y - r = \frac{1}{3}r$, or $\frac{1}{3}$ of the original range.

With three planes, let the first support plane fly x out, then y out, then $y + x$ back, and let the main plane fly x out, then y out, then z out. As before $5x + 3y + z = 3r$. To maximise $x + y + z$ subject to this contraint we must maximise z, so we make $z = r$, and we must then maximise y, so $x + 3y + z = 2r$ (that is, the second support plane and the main plane are both full after the first refuelling). This makes $x = y = \frac{1}{4}r$, so the extension is $x + y + z - r = \frac{1}{2}r$, or $\frac{1}{2}$ of the original range.