2012-2013 Middle Solutions Round 1

M1.

In mathematics, the notation 11! is short for $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
Similarly, another example is, $4! = 4 \times 3 \times 2 \times 1$ and so $4! = 24$.

What is the greatest factor of 11! that is one greater than a multiple of 6?

Note: 11! is spoken of as 'eleven factorial'.

Solution

Mutiples of 6 are even and also divisible by 3 so the answer is an odd number which has no factor of 3.

The odd factors of 11! which are not divisible by 3 are 5, 7 and 11 but, because $10 = 5 \times 2$ there is an extra factor of 5.

So the required number is a factor of $5 \times 7 \times 5 \times 11 = 1925$. This fails as 1924 is not a multiple of 3.

The next biggest factor is $5 \times 7 \times 11 = 385 = 94 \times 6 + 1$.

The answer is 385.

M2.

I take a certain journey and due to heavy traffic crawl along the first half of the complete distance of my journey at an average speed of 10 mph. How fast would I have to travel over the second half of the journey to bring my average speed for the whole journey up to 16 mph? And how fast to bring it up to 20mph?

Solution

Suppose half the total distance is x miles so the first half of the journey takes $\frac{1}{10}x$ hours.

Let the average speed for the second half be n mph so the second half takes $\frac{x}{n}$ hours.

The total journey is $2x$ miles, at an average of 16 mph takes $\frac{2x}{16}$ hours. So $\frac{x}{10} + \frac{x}{n} = \frac{2x}{16}$. So $\frac{1}{n} = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}$.

So I need to travel at an average of 40 mph for the second half.

It is impossible to bring the average speed up to 20 mph as that would take the same amount of time in total as has already been used for the first half.

M3.

All dragons have a head, a body and a tail. A mother dragon and her son were lying in the sun. They noticed that the length of the young dragon was exactly the same as the length of his mother's tail. The length of the body of the mother was three times as long as her own head which was twice as long as the head of her son. The younger dragon's body was 2 feet longer than his head and his tail was one-third of the length of the tail of his mother. When the lengths of the two dragons are added, the total is 48 feet.

Determine the lengths of the head, body and tail of each dragon.

Solution

Let the lengths of head, body and tail of mother and son be H, B, T and h, b, t respectively. Then

$$T = h + b + t, \qquad B = 3H, \qquad H = 2, \qquad b = h + 2, \qquad 3t = T$$

and

$$H + B + T + h + b + t = 48. \tag{$*$}$$

Expressing B, h and b in terms of H,

$$B = 3H, \qquad h = \tfrac{1}{2}H, \qquad b = \tfrac{1}{2}H + 2,$$

giving

$$T = 3t = h + b + t = \tfrac{1}{2}H + \left(\tfrac{1}{2}H + 2\right) + t = H + 2 + t$$

so

$$2t = H + 2, \qquad t = \tfrac{1}{2}H + 1, \qquad T = 3 \times \tfrac{1}{2}(H + 2).$$

Now using (*), $H + B + T + h + b + t = 48$, so

$$H + 3H + \tfrac{3}{2}(H + 2) + \tfrac{1}{2}H + \tfrac{1}{2}H + 2 + \tfrac{1}{2}(H + 2) = 48$$

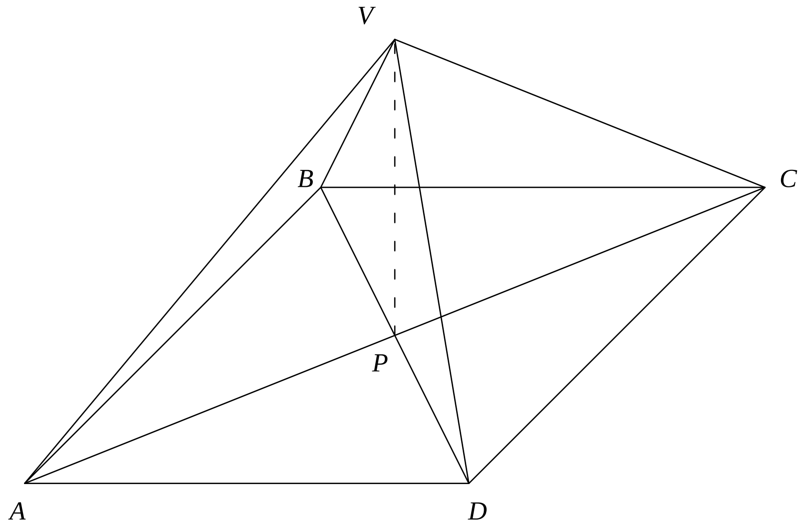
$$\therefore \qquad 7H = 42.$$

This gives the mother's head, body and tail as 6, 18 and 12 feet and the son's as 3, 5 and 4 feet.

M4.

Four spheres each of radius 10 cm lie on a horizontal table so that the centres of the spheres form a square of side 20 cm. A fifth sphere of radius 10 cm is placed on them so that it touches each of the spheres without disturbing them. How far above the table is the centre of the fifth sphere?

Solution



In the diagram, A, B, C, D and V are the centres of the 5 spheres. The shape $ABCDV$ is a square-based pyramid, with all 8 of its edges equal, in this case each edge is 20 cm.

The base is a square so $\triangle ABC$ is a right-angled triangle and we can use Pythagoras' Theorem to get

$$AC^2 = 20^2 + 20^2 = 2 \times 400$$

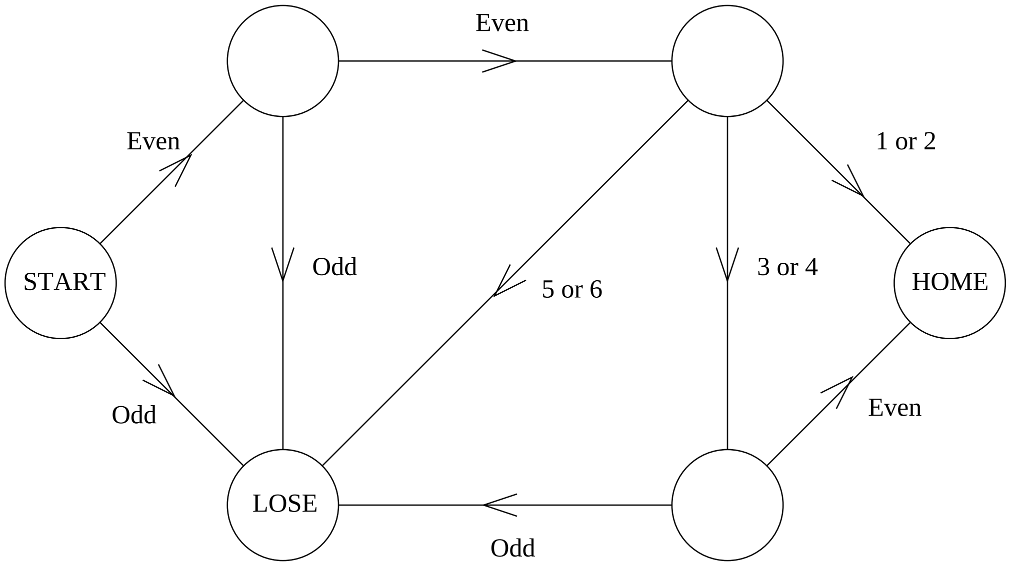
$$\Rightarrow AC = 20\sqrt{2}.$$

But, since the sides of $\triangle VAC$ are 20, 20, $20\sqrt{2}$, we can apply the converse of Pythagoras' Theorem to prove that $\triangle VAC$ is a right-angled triangle. As this triangle is congruent with, for example, $\triangle ABC$, we can say that the height VP is equal to half of a diagonal of the base. Thus we get

$$VP = 10\sqrt{2}.$$

But, the plane $ABCD$ is 10 cm above the table so the height of the centre of the fifth sphere above the table is $10\left(1 + \sqrt{2}\right)$ cm.

M5.



A counter placed on the start circle moves in the direction determined by the throw of a normal six-sided die. What is the probability of reaching the HOME circle?

Solution

There are two routes from START to HOME.

For the fast track, the probability is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$.

For the other track, the probability is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$.

The total probability is therefore $\frac{1}{12} + \frac{1}{24} = \frac{1}{8}$.