2011-12 Middle Set 2 solutions

M1. A set of cards, numbered from 1 to 19, are placed face down on a table. Nine players each pick up two cards. The remaining card is then turned over. The player who achieves the highest total with their two cards plus the number on the remaining card is the winner.

Is it possible for all nine players to have the same total?

If so, what can this total be?

Explain your reasoning.

Solution

The total of the values 1 to 19 is 190. Let n be the centre value. This value will be used a total of nine times (to form the nine possible triples). An expression for the total of all nine triples is the total of the 19 cards plus the duplications of the centre card, i.e. $8n + 190$. Since $8n + 190$ may be written as $9n + 189 - n + 1 = 9(n + 21) + 1 - n$, $1 - n$ has to be divisible by 9. This gives values of n of 1, 10, 19.

n	1	10	19
$8n + 190$	198	270	342
Totals	22	30	38

Each of these leads to possible solutions:

Centre card	Total	Other pairs
1	22	(2,19) (3,18) (4,17) (5,16) (6,15) (7,14) (8,13) (9,12) (10,11)
10	30	(1,19) (2,18) (3,17) (4,16) (5,15) (6,14) (7,13) (8,12) (9,11)
19	38	(1,18) (2,17) (3,16) (4,15) (5,14) (6,13) (7,12) (8,11) (9,10)

Yes, it is possible for all nine players to have the same total and there are three totals $22,\ 30,\ 38$ as shown.

M2. A computer whizz claims that his program has found some numbers which satisfy Fermat's equation $x^n + y^n = z^n$ for a large integer n.

He tells his 10 year old brother that

$x = 31415926536$ $y = 89173261421$ $z = 90354441655$

Almost immediately his brother says that there cannot be any value of n which will work for these numbers. The computer whizz checks his program and finds a bug.

How did his brother know there was a bug?

Solution

We consider the values of the final digit in the power of each number.

Any power of a number ending in 6 also ends in 6 and so x^n ends in 6.

Any power of a number ending in 1 also ends in 1 and so y^n ends in 1.

So the total of x^n and y^n must end in 6 + 1 = 7.

Any power of a number ending in 5 also ends in 5 and so z^n ends in 5.

Thus the last digits of the two sides of the equation do not match, and so the numbers given must be incorrect.

M3.

1	2	3
4	5	6
7	8	9

Take a pair of numbers from the numbers in the grid where the first one is immediately below the second, for example 8 and 5. Use the numbers to form a calculation as shown:

$$85 - 58 = 27.$$

Using another pair, say 4 and 1, we have

$$41 - 14 = 27.$$

Show that the answer is always 27, and explain why this happens.

Solution

The array is of the form

$$\begin{array}{ccc} a & a+1 & a+2 \\ a+3 & a+4 & a+5 \\ a+6 & a+7 & a+8 \end{array} \qquad (\text{with } a = 1)$$

The numbers chosen can thus be expressed in the form $a + d$, $a + d + 3$. Then

$(10(a + d + 3) + (a + d) - (10(a + d) + (a + d + 3)) =$

$$10a + 10d + 30 + a + d - 10a - 10d - a - d - 3 = 27.$$

M4. Catriona went on a day trip to the land of Silverglade where the money is very colourful. She exchanged her money for 29 knomes which came as one red note, one yellow note, one blue note and one green note. Her guide pointed out that she now had one of each of the notes available in Silverglade but that many shops did not take the green notes.

At a bank, Catriona exchanged her green note for a blue and a yellow note. At the shop next door she bought a potion for 1 knome and, having handed over two yellow notes, she got three red notes in change. Later, she paid for a postcard costing 1 knome with two blue notes and she got three yellow notes in change.

What is the value of each of the four different coloured notes in Silverglade?

Solution

We use the initial letter to represent the value of each note.

So Catriona has:

$$R + Y + B + G = 29 \tag{1}$$

Exchange:

$$G = B + Y \tag{2}$$

Potion:

$$2Y - 3R = 1 \tag{3}$$

Postcard:

$$2B - 3Y = 1 \tag{4}$$

Using (1) and (2)

$$R + 2B + 2Y = 29$$

Using (4)

$$R + 3Y + 1 + 2Y = 29$$

$$R + 5Y = 28$$

Using (3)

$$R + 2\tfrac{1}{2}(3R + 1) = 28$$

$$17R = 51$$

$$R = 3$$

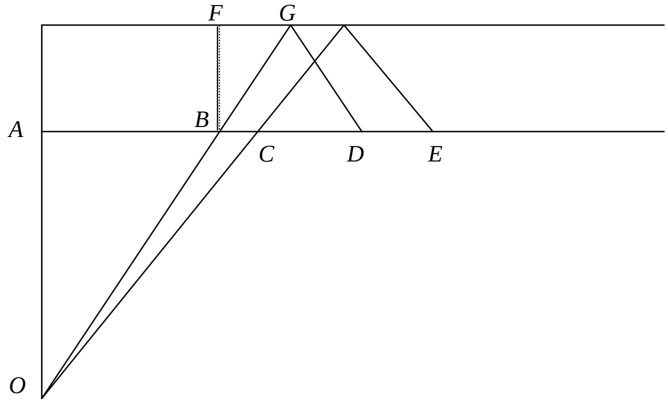
The value of each of the notes is $R = 3$, $Y = 5$, $B = 8$ and $G = 13$.

M5. Andy is standing at a bus stop near his house. Through a small window, he can see the reflection of a television in a large mirror. The television set is mounted on the same wall of the house as the window and the mirror is on the opposite wall. He also notices that the reflection he sees through the small window is the full width of the TV but no more.

He wonders how wide his neighbour's TV is. But as the house is exactly like his own he can work it out. The small window is 50 cm wide and the room is 4 m deep. Furthermore he is exactly 10m from the nearest point on the front wall of the house on which the window and the TV are. How does he calulate this and what is the width of the TV?

Solution

We draw a diagram to represent the situation and solve the problem.



In this diagram, Andy is at O, the window is BC and the TV is DE. Note that $OA = 10$ m and $BF = 4$ m.

Let the length of AB be x m.

By similar triangles, $\dfrac{FG}{FB} = \dfrac{AB}{OA}$ so $FG = \dfrac{4}{10}x$.

So $BD = 2 \times FG = \dfrac{8}{10}x$ and $AD = x + \dfrac{8}{10}x = \dfrac{18}{10}x$.

Note that $AC = x + \frac{1}{2}$ and exactly the same method gives that $AE = \dfrac{18}{10}\left(x + \dfrac{1}{2}\right)$.

Thus the width of the TV is $DE = AE - AD = \dfrac{9}{10}$, i.e. the TV is 0.9 m wide.