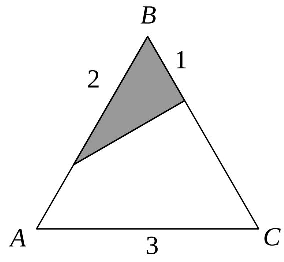
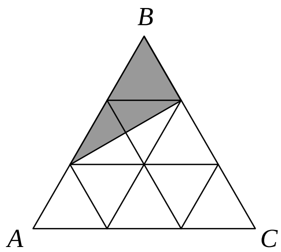
2011 Middle Set 1 solutions

M1. The triangle *ABC* is equilateral. What fraction of it is shaded?
Explain your reasoning.



Solutions

Method 1



We can split the equilateral triangle into 9 equilateral triangles with side length 1.
The two small shaded triangles cover half of a rhombus formed from 2 triangles so their area is 1.
The total shaded area is the same as 2 triangles.
The area of the shaded part would be $\frac{2}{9}$ of the large triangle.

Method 2

There is neat solution using the trigonometric formula for the area of a triangle.
First note that as the outer triangle is equilateral, all its sides have length 3.
Thus

$$\frac{\text{shaded area}}{\text{whole triangle}} = \frac{\frac{1}{2} \times 1 \times 2 \times \sin 60^\circ}{\frac{1}{2} \times 3 \times 3 \times \sin 60^\circ} = \frac{1 \times 2}{3 \times 3} = \frac{2}{9}.$$

M2. Six cousins have a get together and discover that the mean of their ages is 19. Four of the ages are 21, 19, 23 and 11 and range of all six ages is 20. Find the other two ages. **Explain your reasoning.**

Solution

There are 6 cousins. Since their mean is 19 the total of their ages is 6×19 which is 114.

The total $21 + 19 + 23 + 11 = 74$.

Let the missing numbers be a and b. From above it follows that $a + b = 40$.

Since the range is 20 if 23 is the largest then the smallest must be 3 which means that the other missing number must be 37. But 37 would then be the largest not 23.

If 11 was the smallest then the largest would be 31 which means the other missing number is 9. However, this would then be the smallest not 11.

This means that a and b must be the largest and smallest numbers in the set giving:

$$a + b = 40$$

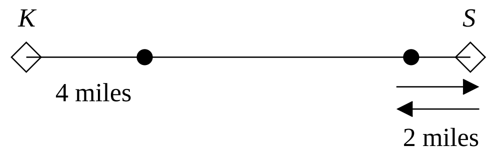
$$a - b = 20$$

Therefore the missing numbers are 30 and 10.

M3. Andrew set out to cycle from Kirkton to Simsburgh at exactly the same time as Mike left Simsburgh to cycle to Kirkton, along the same route. The two passed each other at the point on the route which is 4 miles from Kirkton. Each cyclist continued to his destination, turned and immediately started cycling back towards his starting point at the same rate. This time, they met 2 miles from Simsburgh.

Assuming that Andrew and Mike travelled at constant speeds, how far is it from Kirkton to Simsburgh? Given that Andrew's speed is v mph, what is Mike's speed?

Solution



When they first met, the total distance travelled by the 2 cyclists was KS.
When they met the second time, the combined distance travelled was $3 \times KS$.

Since both travelled at constant speed, when they met the second time, they had both travelled 3 times as far as at their first meeting.

That is, Andrew had travelled 12 miles which took him to the point 2 miles from S. Hence $KS = 10$ miles.

At the first meeting, Andrew had travelled 4 miles while Mike travelled 6 miles. They travelled at constant speeds so Mike's speed was $1.5v$ mph

M4. The rectangular floor of a room is completely tiled with whole tiles, each of which is 15 cm square. The black tiles form a border of width one tile round the room. Within the black border all the tiles are red. There are exactly twice as many red tiles as black ones. Determine the possible lengths and corresponding widths for the room.

Solution

Let the number of tiles along the length of the room be m.

Let the number of tiles across the width of the room be n.

Then there are $2\left[(m-1)+(n-1)\right]$ black tiles in the border and $(m-2)(n-2)$ red tiles in the centre.

We require

$$\begin{aligned}(m-2)(n-2) &= 2\times 2\left[(m-1)+(n-1)\right]\\ mn-2m-2n+4 &= 4m+4n-8\\ mn-6m-6n+12 &= 0\\ (m-6)(n-6)-36+12 &= 0\\ (m-6)(n-6) &= 24\end{aligned}$$

The possible values for $m-6$ and $n-6$ are given in the table. (Note that $m > n$.)

$m-6$	$n-6$	m	n	length metres	width metres
24	1	30	7	4.5	1.05
12	2	18	8	2.7	1.2
8	3	14	9	2.1	1.35
6	4	12	10	1.8	1.5

There are 4 possible room dimensions, as given in the table above.

(The room is more of a corridor or a store cupboard!)

M5. Show that there is only one set of different positive integers, x, y, z, such that

$$1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

i.e. 1 can be expressed as the sum of the reciprocals of three different positive integers in only one way.

Deduce that, if n is any odd integer greater than 3, then 1 can be expressed as the sum of n reciprocals of different positive integers.

For which even integers is this possible? **Justify your answer**.

Solution

Since the integers must be different, let $x < y < z$.

x cannot be 1 because the left-hand side value 1 would then be reached.

If x is 2, then y must be 3 or more. If y is 3, then

$$\frac{1}{z} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and so one solution is

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

If $y = 4$, then z must be 5 or more, and the total of 1 would never be reached.

If x is 3, then y must be at least 4 and z at least 5. Hence again the total of 1 would never be reached.

If x is greater than 3, then y must be greater than 4 and z greater than 5, so that the shortfall would be even greater.

Thus there is only the one solution already found.

Moving to odd numbers of terms in excess of 3.

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \end{aligned}$$

Multiplying the last and smallest reciprocal by $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ has increased the number of reciprocals by 2, and this can be done repeatedly to obtain any odd number of different reciprocals with sum 1.

For just 2 reciprocals, let $1 = \frac{1}{x} + \frac{1}{y}$ with $x \leqslant y$.

If $x = 2$, then y must also equal 2. But then x and y are not different.

If x is 3 or more then y must be greater than or equal to 3 and the total can never reach 1.

Hence there is no solution for 2 reciprocals.

For 4 reciprocals

$$
\begin{aligned}
1 &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}.
\end{aligned}
$$

This satisfies the criteria and the number of reciprocals can be increased by 2 by multiplying the last and smallest reciprocal by $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ as before to obtain any even number of different reciprocals with sum 1.

Thus the only number of terms which cannot be used is 2.