2010-11 Middle Set 2 solutions

M1. Four cards, each numbered with a different whole number, are placed face down. Four people, Gavin, Jack, Katie and Luke, in turn select two of these cards, write down their total, and then replace the two cards. Gavin's total is 6, Jack's 9, Katie's 12 and Luke's 15.

Two of the cards are then turned over and their total is 11.

Determine the numbers on each of the cards.

Solution

Let the values of the four cards $a, b, c,$ and d.

There are 6 possible combinations of two cards which can be selected

$$\begin{array}{cc} a, b & c, d \\ a, c & b, d \\ a, d & b, c \end{array} \qquad (1)$$

These can be combined to give three combinations of a, b, c, d, all of which have the same total.

$$a, b, c, d \qquad a, c, b, d \qquad a, d, b, c$$

Now we know that 5 of the six pairs in (1) give totals of 6, 9, 12, 15, 11. Taking combinations of these pairs

$6 + 9 = 15$	$9 + 12 = 21$	$12 + 15 = 27$	$15 + 11 = 26$
$6 + 12 = 18$	$9 + 15 = 24$	$12 + 11 = 23$	
$6 + 15 = 21$	$9 + 11 = 20$		
$6 + 11 = 17$			

As 21 is the only total which appears twice then $a + b + c + d = 21$ and the missing total is $21 - 11 = 10$.

Suppose that a, b, c, d are listed in order then

$$\begin{array}{rcl} a + b &=& 6 \\ a + c &=& 9 \\ a &=& 6 - b \end{array} \qquad \begin{array}{rcl} c + d &=& 15 \\ b + d &=& 12 \\ d &=& 12 - b \end{array}$$

So,

$$(6 - b) + c = 9 \qquad c + (12 - b) = 15.$$

Hence $c - b = 3$ and either

$b + c = 10$ (not possible) or $b + c = 11$ (so $c = 7$ and $b = 4$).

This leads to $a = 2$ and $d = 8$.

The numbers on the cards are: 2, 4, 7, 8.

M2. An old-fashioned rectangular billiard table has only four pockets, one at each corner. The lengths of the sides of the table form a whole number ratio.

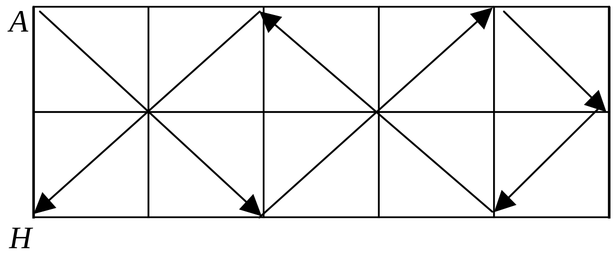
Show that, if the ratio is 5 : 2 and a ball is hit from one corner at an angle of 45°, it will land in a pocket after 5 rebounds.

If the ratio of the sides were $m : n$, where m and n are different whole numbers, with no common factor, and the ball were hit from a corner at an angle of 45°, show that the ball would always drop into a pocket after a number of rebounds. How many rebounds would there be in this case?

Solution

For a 5×2 table, starting at A, we have

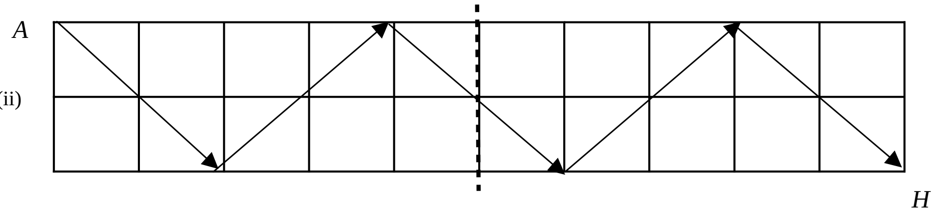
fig (i)



number of rebounds = 5.

Note that the movement could be conveniently represented on a 10×2 grid, as below, where the right half, when folded back over the left half, shows the movement from right to left.

fig (ii)



For any $m \times n$ table, using a diagram as in fig (ii) with a $2m \times n$ grid, there would be $n - 1$ sections lying over the first and there would be an impact on each, giving $n - 1$ impacts on the ends of the table; similarly, from, fig (ii), there will be m movements of the ball to take it across the grid, all except the last ending in a bounce so there are $m - 1$ rebounds on the long edges. Hence the total number of rebounds is $m + n - 2$.

M3. A farmer was having cash-flow problems and was discussing his options with his wife. "If we sell 75 chickens we will bring in some money and my existing stock of feed will last an extra 20 days. But if we buy an additional 100 chickens, we will get money from the extra eggs, but my existing stock of feed will last 15 days less". "Exactly how many chickens do you currently have?" asked his wife.

What is the answer to his wife's question and why is this the answer?

Solution

Suppose the farmer has N chickens and enough feed to last them M days. So he has NM daily chicken feeds altogether. So $NM = (N - 75)(M + 20)$ and $NM = (N + 100)(M - 15)$. From these two equations we get

$$-75M + 20N = 1500$$

$$100M - 15N = 1500.$$

Hence

$$-300M + 80N = 6000$$

$$300M - 45N = 4500.$$

Adding these gives

$$35N = 10500$$

$$N = 300$$

Thus the farmer initially had 300 chickens.

M4. The shape of a fifty-pence piece is based on a regular heptagon which is a 7-sided polygon. The distance between each vertex and each of its two 'nearly opposite' vertices is 1 unit. The perimeter of the coin is formed by circular arcs of radius 1 unit which are centred on each vertex, and join the two nearly opposite vertices. Find the length of the perimeter of the coin.

Solution

The arc between two neighbouring vertices subtends an angle of $\frac{1}{7} \times 360°$ at the centre of the circumscribing circle.

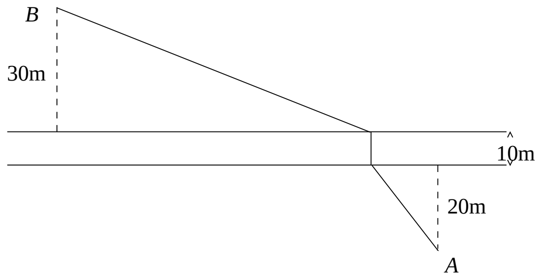
The inscribed angle is half of the central angle (the angle at the centre of a circle is twice the angle at the circumference), so the arc between two neighbouring vertices subtends an angle of $\frac{1}{7} \times 180°$ at the opposite vertex.

Thus the arc length between a neighbouring pair of vertices is $\frac{1}{7} \times$ circumference of a circle of radius $\frac{1}{2} = \frac{1}{7} \times \left(2\pi \times \frac{1}{2}\right) = \frac{1}{7}\pi$.

There are 7 arcs, so the total perimeter is $7 \times \frac{1}{7}\pi = \pi$.

The perimeter of the coin is π.

M5.



A rabbit's burrow is at *A* and he knows that there are carrots in a garden at *B*, across a road, which is 10m wide. The burrow is 20m from the nearer edge of the road and the carrots are 30m beyond the other edge as shown in the diagram. The straight line distance from *A* to *B* is 80m.

The rabbit is wary of crossing the road and knows from past experience that he must cross directly across the road, not askew. What is the length of the shortest possible route for the rabbit from the burrow to the carrots?

Solution

Since the rabbit always crosses straight across the road, any route he takes will be 10m plus the distance from *A* to a point on one edge of the road plus the distance from the opposite point on the road to *B*. But this amounts to removing the road and considering the shortest distance then from *A* to *B*, which is, of course, a straight line.

To obtain this we use Pythagoras theorem. On the original diagram, it is 80m from *A* to *B*. So the 'horizontal' distance between *A* and *B* is $\sqrt{80^2 - 60^2}$.

With the road 'removed', the new distance between *A* and *B* is

$\sqrt{50^2 + (80^2 - 60^2)} = 10\sqrt{53}$ metres.

So the shortest distance the rabbit has to travel is $10 + 10\sqrt{53}$ metres.