2010 Middle Set 1 solutions

M1. Amanda, Brian and Claire enter the school talent contest. They each perform in one of three rooms in the morning and in a different one of the three rooms in the afternoon.

We know that

- Amanda's act is maths magic,
- one pupil moves from the hall to the gym,
- Claire is in the drama studio after lunch,
- Brian's morning room is taken by the singer in the afternoon,
- one pupil's act is juggling.

Find out where each person performs in the morning and in the afternoon, and what their act is.

Justify your answer.

Solution

The initial information is:

	Morning	Afternoon	Act
Amanda			Maths Magic
Brian			
Claire		Drama Studio	

Brian's morning room is taken by the singer in the afternoon tells us that he is not the singer so Claire must be the singer and Brian must be the juggler. It also tells us that Brian was in the Drama Studio in the morning.

	Morning	Afternoon	Act
Amanda			Maths Magic
Brian	Drama Studio		Juggler
Claire		Drama Studio	Singer

Amanda must move from the hall to the gym. So we now know that Claire was in the gym in the morning and Brian was in the hall in the afternoon.

	Morning	Afternoon	Act
Amanda	Hall	Gym	Maths Magic
Brian	Drama Studio	Hall	Juggler
Claire	Gym	Drama Studio	Singer

M2. Write down any whole number containing four digits. Now write down a second number containing the same digits in a different order. Show that, when you take the smaller number from the larger number, you obtain a multiple of 9.
Explain why this always works for any four-digit number.

Solution
Numerical example: 1234; reverse 4321
Difference = $3087 = 9 \times 343$

Consider any four digits a, b, c and d and form the two numbers

$$abcd \qquad \text{and} \qquad cadb.$$

Written out, the numbers are

$$1000a + 100b + 10c + d \qquad \text{and} \qquad 1000c + 100a + 10d + b.$$

Subtracting these numbers gives $900a + 99b - 990c - 9d$ which is divisible by 9.
No matter how the digits are rearranged, the same pattern will occur.

Alternative for the general case:
Consider $1000a + 100b + 10c + d$ and rearrange it

$$\begin{aligned} 1000a + 100b + 10c + d &= 999a + 99b + 9c + (a + b + c + d) \\ &= 9(111a + 11b + c) + (a + b + c + d) \end{aligned}$$

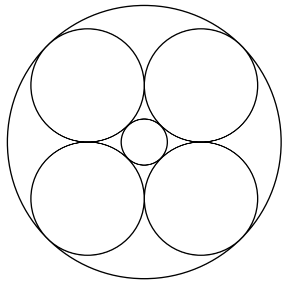
Thus any 4-digit number is expressible as a multiple of 9 plus the sum of its digits.
In the present situation, we have two 4-digit numbers with the same digit sum so their difference is the difference of two multiples of 9 and so must be a multiple of 9.

M3. In the diagram there are six circles: one small, four medium and one large, touching as shown.

The radius of each of the medium circles is 1cm.

What is the radius of the large circle?

What is the radius of the small circle?



Solution

In the diagram, let O be the centre of both the large circle and the small circle and let B be the centre of a medium circle.

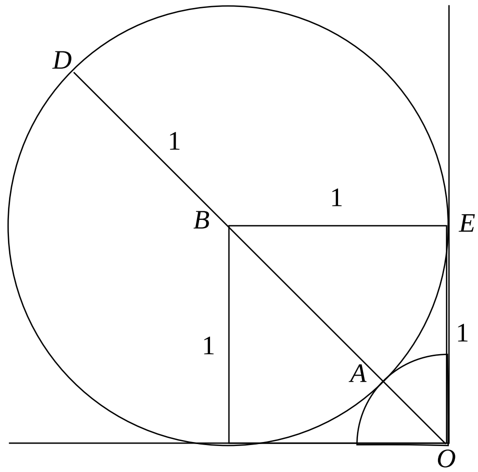
From the triangle OBE, $OB = \sqrt{2}$

Radius of large circle =

$$OD = OB + BD$$
$$= \sqrt{2} + 1$$

Radius of small circle =

$$OA = OB - AB$$
$$= \sqrt{2} - 1$$



M4. Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each colour did she have?

Solution

Suppose Katie had r red beads, g green beads and b blue beads. Then

$$r(r + g) = 120 + b.$$

If $b = 2$, then the right-hand side is $122 = 2 \times 61$ and so $r = 2$. But the numbers of beads are all different. So b must be an odd prime.

This means that the right-hand side is odd. Thus both r and $r + g$ must be odd, so g must be even and prime.

So $g = 2$ and the equation becomes $r^2 + 2r = 120 + b$.

Hence $b = r^2 + 2r - 120 = (r - 10)(r + 12)$.

Since b is prime, $r - 10 = 1$ so $r = 11$ and $b = 23$.

Thus Katie had 11 red beads, 2 green beads and 23 blue beads.

M5. Ant and Dec had a race up a hill and back down by the same route. It was 3 miles from the start to the top of the hill. Ant got there first but was so exhausted that he had to rest for 15 minutes. While he was resting, Dec arrived and went straight back down again. Ant eventually passed Dec on the way down just half a mile before the finish.

Both ran at a steady speed uphill and downhill and, for both of them, their downhill speed was one and a half times faster than their uphill speed. Ant had bet Dec that he would beat him by at least a minute.

Did Ant win his bet?

Explain your answer.

Solution

Let Ant's uphill speed be *a* mph and Dec's be *b* mph.

Suppose that Ant had been resting for *x* hours when Dec arrived (where *x* is between 0 and $\frac{1}{4}$). Then, calculating their times to the top of the hill and then until Ant passed Dec on the way down we have:

Time going up $$\frac{3}{a} + x = \frac{3}{d},$$

and the time going down $$\frac{\frac{5}{2}}{\frac{3}{2}a} + \left(\frac{1}{4} - x\right) = \frac{\frac{5}{2}}{\frac{3}{2}d}.$$

So rearranging each of these:

$$\frac{1}{d} - \frac{1}{a} = \frac{x}{3} \text{ and } \frac{1}{d} - \frac{1}{a} = \frac{3}{5}\left(\frac{1}{4} - x\right).$$

$$\frac{x}{3} = \frac{3}{5}\left(\frac{1}{4} - x\right)$$

$$\frac{5x}{9} = \frac{1}{4} - x$$

$$20x = 9 - 36x \Rightarrow x = \frac{9}{56}.$$

Dec's time for the whole race minus Ant's time for the whole race =

$$\left(\frac{3}{d} + \frac{3}{\frac{3}{2}d}\right) - \left(\frac{3}{a} + \frac{1}{4} + \frac{3}{\frac{3}{2}a}\right) = 5\left(\frac{1}{d} - \frac{1}{a}\right) - \frac{1}{4} = \frac{1}{5} \times \frac{9}{56} - \frac{1}{4} = \frac{1}{56} > \frac{1}{60}.$$

So Ant did beat Dec by more than a minute and won his bet.