2009 Middle Set 2 solutions

M1. How many squares, of *all* sizes, are seen on a standard, 9×9 Sudoku grid?

Solution

Consider the numbers of squares of all possible sizes:

1×1 squares	there are $9 \times 9 = 81$
2×2	these could have their top left corner in any square which is not in the last row or column giving $8 \times 8 = 64$ possibilities
3×3	$7 \times 7 = 49$

etc

Hence the number of squares is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 = 285$.

M2. I took the same route when I cycled to and from work today. Due to a strong headwind on the way home, the return journey took twice as long as the journey to work. If my average speed for the whole journey was 16 km/h what were my average speeds to and from work?

Solution

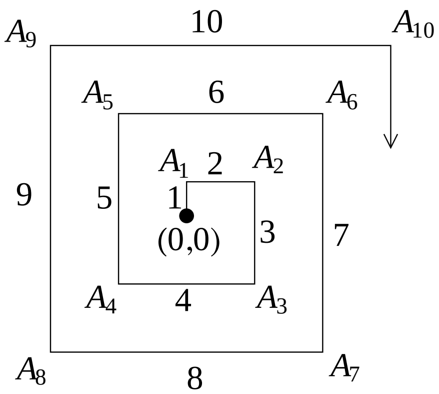
Let the distance to work be x km and the journey time in the morning be t hours.

Total time = $3t = \frac{2x}{16} = \frac{x}{8}$ so that $\frac{x}{t} = 24$ and hence the speed on way to work = 24 km/h.

On the return journey, average speed = $\frac{x}{2t} = \frac{1}{2}\frac{x}{t} = 12$ so the return journey speed is 12 km/h.

M3. The diagram below illustrates a spiral sequence A_0, A_1, A_2, A_3, ... which starts at the origin. The first leg of the spiral has length 1 and after the that, the length increases by one each time.

What is the sum of the 6 coordinates of A_{2001}, A_{2002} and A_{2003}?



Solution

Looking at the coordinates of the points A_0, A_1, …

$A_0(0,\ 0)$	$A_1(0,\ 1)$	$A_2(2,\ 1)$	$A_3(2,\ -2)$	
$A_4(-2,\ -2)$	$A_5(-2,\ 3)$	$A_6(4,\ 3)$	$A_7(4,\ -4)$	
$A_8(-4,\ -4)$	$A_9(-4,\ 5)$	$A_{10}(6,\ 5)$	$A_{11}(6,\ -6)$	etc

Note that the points A_0, A_4, A_8, …, where the suffix is a multiple of 4 are given by $A_n\left(-\frac{n}{2},\ -\frac{n}{2}\right)$.

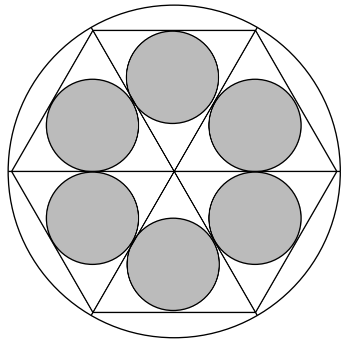
Also, when n is a multiple of 4:

$$A_{n+1} \text{ is } \left(-\frac{n}{2},\ \frac{n}{2} + 1\right);\ A_{n+2} \text{ is } \left(\frac{n}{2} + 2,\ \frac{n}{2} + 1\right) \text{ and } A_{n+3} \text{ is } \left(\frac{n}{2} + 2,\ -\frac{n}{2} - 2\right).$$

Taking $n = 2000$, we get $A_{2000}\,(-1000, -1000)$, $A_{2001}\,(-1000,\ 1001)$, $A_{2002}\,(1002,\ 1001)$ and $A_{2003}\,(1002, -1002)$.

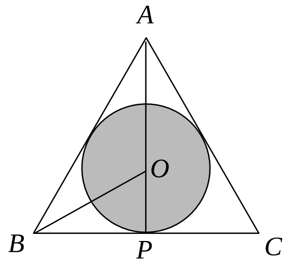
So the required sum is: $-1000 + 1001 + 1002 + 1001 + 1002 - 1002 = 2004$.

M4. The diagram shows a regular hexagon with its diagonals drawn and six circles fitted into the regions created. What fraction of the circumscribing circle is shaded?



Solution

Consider one of the shaded circles.



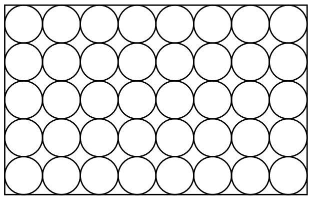
Let $OP = r$ = radius of the small circle and $AB = BC = CA = R$ = radius of the large circle. Then $AB^2 = AP^2 + BP^2$. So $R^2 = AP^2 + \left(\frac{1}{2}R\right)^2$. Thus $AP^2 = \frac{3}{4}R^2$. Since $OB = OA$, $AP = r + OA = r + OB$, i.e. $OB = AP - r$. Now, applying Pythagoras' Theorem to $\triangle OBP$, we get

$$\begin{aligned} OB^2 &= BP^2 + OP^2 \\ (AP - r)^2 &= \left(\tfrac{1}{2}R\right)^2 + r^2 \\ AP^2 - 2AP.r + r^2 &= \tfrac{1}{4}R^2 + r^2 \\ \tfrac{3}{4}R^2 - \tfrac{2\sqrt{3}}{2}Rr &= \tfrac{1}{4}R^2 \\ \tfrac{1}{2}R^2 &= \sqrt{3}Rr \\ R &= 2\sqrt{3}\,r \end{aligned}$$

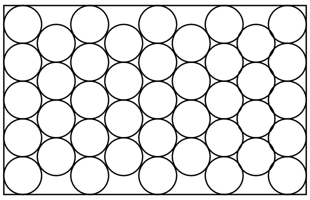
Hence, the area of the circumcircle is $\pi R^2 = 12\pi r^2$ which is exactly twice the shaded area.

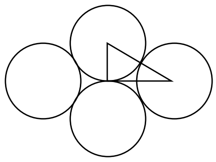
Alternative Solution. Area of circumscribed circle is πR^2. Shaded area is $6\pi r^2$ where $r = OP$. From the triangle OBP, $BP = \frac{1}{2}BC = \frac{1}{2}AB = \frac{1}{2}R$. So $r / \left(\frac{1}{2}R\right) = \tan 30° = 1/\sqrt{3}$. So $r = \dfrac{R}{2\sqrt{3}}$. So shaded area is $6 \times \frac{r^2}{12} = \frac{1}{2}r^2$.

M5. Forty cylindrical tubes, each with diameter one inch and equal lengths, are packed as shown snugly in 5 rows of 8 each in a box so that they may be transported without rattling. Show that the box could be repacked with forty-one of the same sized cylindrical tubes. Will they now rattle?



Solution





The box can be repacked as shown.
Consider a cluster of four circles. The right-angle triangle shown has known sides $\frac{1}{2}$, 1. Hence by Pythagoras' Theorem, the third side is $\sqrt{1^2 - \frac{1}{2}^2} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$.
This represents the distance between the centres of adjacent columns of which there are 9. Hence the total distance from left to right is $\frac{1}{2} + \frac{8}{2}\sqrt{3} + \frac{1}{2} = 4\sqrt{3} + 1$ and this is less than 8.
Thus they can be packed in the box.
Since $4\sqrt{3} + 1$ is strictly less than 8 there will be room for the tubes to rattle in the box.