2009 Middle Set 1 solutions

M1 Mr and Mrs McLeod have six children – Andrea, John, Eilidh, Rory, Fiona and Pat. Just before Christmas, the six children went on their own to town to do some shopping. They all spent some money, each spending a whole number of pounds. When they returned home, they told their parents, in a roundabout way, how much they had spent as follows: Andrea and John together had spent £26, Eilidh and Rory together had spent £20 and Fiona had spent £9. They didn't say how much Pat had spent, but they did say that one of them had spent £15 more that the average for all the children. Mr and Mrs McLeod thought about this and then started arguing about how much Pat had spent. Explain why they were arguing and say what you can about the amount Pat had spent.

Solution

The total amount of money we are told about is £55 and so the average expenditure is greater than £9. It has to be a whole number since one child spent £15 more than the average and each child spent a whole number of pounds.

So if the child who spent £15 more than the average was not Pat it could only have been either Andrea or John, with one of them spending £25 and the other £1. In that case, the average must have been £10 and the total spent was £60. Thus Pat spent £5.

On the other hand, if it was Pat who spent £15 more than the average, suppose he spent £x. Then $x = 15 + \dfrac{55 + x}{6}$. Thus $x = 29$, i.e. Pat spent £29.

They argued because there are two possible solutions to this problem.

M2 One of the tiles in a floor has the shape of a regular polygon. If the tile is removed from the floor and rotated through 50°, it will fit exactly into its original place on the floor. What is the least number of sides that the tile can have?

Solution

Let the polygon have n sides, then a rotation through 360° moves each side through n places.

Suppose a rotation through 50° moves each side round k places and so a rotation of $\dfrac{50}{k}$ degrees moves each side round 1 place. Then a rotation through $n \times \dfrac{50}{k}$ degrees moves each side round n places. So $n \times \dfrac{50}{k} = 360$. So $\dfrac{k}{n} = \dfrac{50}{360} = \dfrac{5}{36}$ with this fraction in its lowest terms.

So the smallest number of sides is 36.

M3 Six years ago, an investor put a certain sum of money into stocks and shares. At the end of five years, the stocks and shares showed a total growth of 40% but in the sixth year they lost 30% of their value. At the same time, the investor put the same amount of money into a bank account giving 5% per annum for the first five years. In the sixth year, the bank account only gave 2.5%. Having left the bank interest to accrue over the six years, at the end of the six years, the total value of the stocks and shares and the money in the bank was slightly over £288 more than the amount initially invested. How much money did the investor initially put into stocks and shares?

Solution

Suppose the initial investment in the stocks and shares was £x.

After 5 years, they were worth £$(1.4x)$ and after 6 years they were worth £$(1.4 \times 0.7)x$ = £$0.98x$.

After 5 years the money in the bank was £$\left(1.05^5 x\right)$ and after 6 years was £$1.05^5 \times 1.025x$. Thus the total amount of increase was

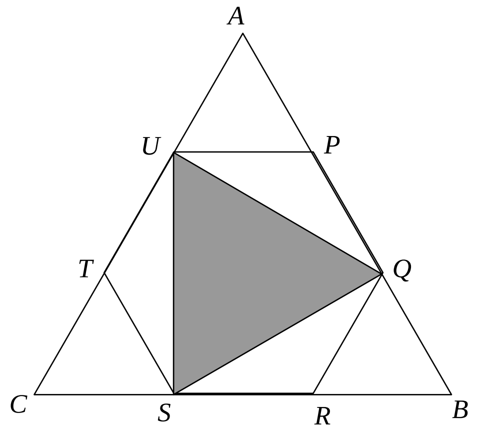
$$\left(0.98 + 1.05^5 \times 1.025 - 2\right)x = 288.$$

This gives $x \approx 1000$. Therefore £1000 is the initial amount invested in stocks and shares.

{A more accurate value for x is 999.346.}

M4 The corners are cut off an equilateral triangle ABC to form a regular hexagon $PQRSTU$ as shown in the diagram.

Find the ratio of the area of triangle UQS to that of triangle ABC.



Solution

Since $PQRSTU$ is a regular hexagon, $AU = AP$ and so APU is an equilateral triangle as all angles are 60°. Also $UP = PQ$ and so $AP = PQ$. So area of triangle APU is equal to the area of triangle PQU having equal bases and height.

Since $PQRSTU$ is a regular hexagon, $\angle SRQ = 120° \Rightarrow \angle QRB = 60°$ as is $\angle RBQ$ so QRB is an equilateral triangle. So $QR = RB$. But $QR = RS$ and so $SR = RB$. So area of triangle QRB is equal to the area of triangle SRQ as they have equal bases and height.

So the area of triangle UQS is equal to the area of triangle ABC less six times the area of triangle QRB. If we divide triangle UQS into three isosceles triangles about its centre point, each of these triangles has the same area as triangle QRB.

But triangle ABC can be divided into nine equilateral triangles identical to QRB so its area is nine times the area of triangle QRB.

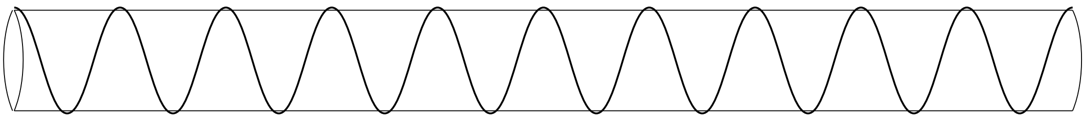
So the area of triangle UQS is one third of the area of triangle ABC.

Alternative:

Superimpose the equilateral triangular lattice on the picture (so ABC is divided into 9 equilateral triangles of 1/3 the height of ABC. Then it is clear that the length US is 2/3 of the height of ABC and the ‘height’ of US from US to Q is 1/2 the base length CB of ABC. Thus the ratio of areas is

$$\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}.$$

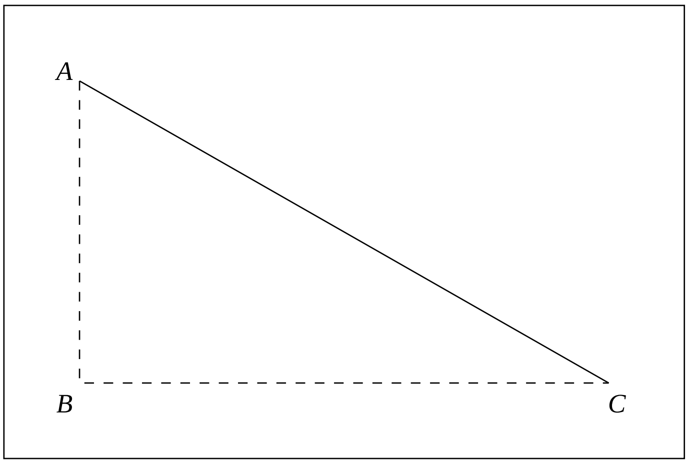
M5



Ten turns of a wire are helically wrapped round a cylindrical tube with outside circumference 4 inches and length 9 inches. At the start and the finish, the end of the wire is at the top. Find the length of the wire.

Solution 1

Consider very carefully fixing one end of the wire on a board and rolling the cylinder until all the wire is on the board.



$BC = 9$ as it is the length of the cylinder. AB is equal to 10 revolutions of the cylinder. Hence

$$AB = 10 \times 4 = 40$$

So, using Pythagoras theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 1600 + 81 = 41^2 \end{aligned}$$

Thus the length of the wire is 41 inches.

Solution 2

The length of the wire will be 10 times as long as the piece of wire wrapped once round a cylindrical tube of circumference 4 inches and length 9/10 inches. Cut this open along the cylindrical element to obtain a rectangle of width 4 inches and height 9/10 inches. The wire is then a diagonal of this rectangle so, by Pythagoras has length in inches $\sqrt{4^2 + \left(\frac{9}{10}\right)^2} = \frac{41}{10}$.

Thus the length of the wire is 41 inches.