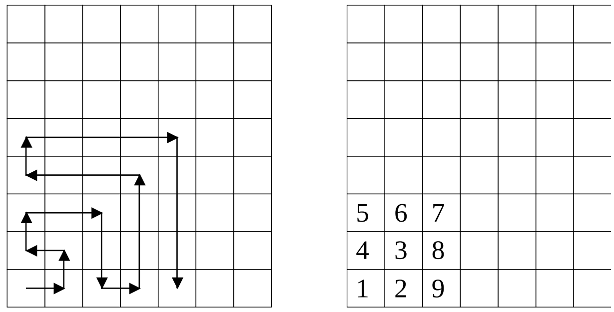
2008-2009 Middle Division Set 2 Solutions

M1. Shaun starts to write down the natural numbers in the square cells of a very large piece of graph paper. He starts at the bottom left corner and writes down the numbers using the following arrangement.



We will identify each of the cells using co-ordinates (x, y) where x is the number of positions to the right and y is the number of position up from the bottom. For example, the cell containing the number 8 has the co-ordinates (3,2).

If N is an even number, what number appears in the cell with co-ordinates $(1, N)$? In what cell does the number 2009 appear? Explain your answers.

Solution

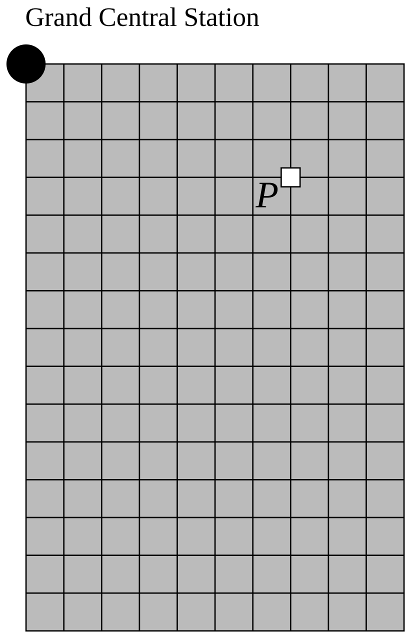
When the path reaches the cell $(1, N)$ for N an even number, it has passed through all cells in the bottom left $N \times N$ block of the graph paper. So the number appearing there will be N^2.

From cell $(1, N)$ the path traverses the cells $(1, N+1)$, $(2, N+1)$, …, $(N+1, N+1)$ along a horizontal row and then down a vertical column traversing the cells $(N+1, N)$, $(N+1, N-1)$, …, $((N+1, 1)$.

Now $2009 = 44^2 + 73 = 44^2 + 45 + 28$ so that 2009 will lie in cell $(45, 45-28) = (45, 17)$.

M2. The street system in New York is built up as a series of blocks. The section in which Gordon works is 10 blocks wide and 15 blocks long and Grand Central Station is located in the top north-west corner of the section. When asked where exactly he worked, he would not specify the location, but said that from Grand Central station, starting on January 1st 2009, he could take a different route to work every day except Christmas Day (which he took off anyway!) but that on the January 1st 2010, he would need to repeat a route already used. If Gordon only walks either south or east, find out where he works.

Give your answer as grid location from the station, for example, *P* is 3 blocks south, 7 blocks east.



Solution

In 2009 there are 365 days but Gordon does not go to work on Christmas Day. So there must be exactly 364 routes from the station to where Gordon works.

The table below shows the number of routes from the station to each point of intersection. (There is just one route to each intersection which is on an edge, thereafter, the number of routes is the sum of the number above and the number to the left - similar to Pascal's triangle.)

blocks	*east*	1	2	3	4	5	6	7	8	9	10
south	station	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	506
5	1	6	21	56	126	252	462	792			
6	1	7	28	84	210	462					
7	1	8	36	120	330						
8	1	9	45	165	495						
9	1	10	55	220							
10	1	11	66	286							
11	1	12	78	**364**							
12	1	13	91								
13	1	14	105								
14	1	15									

It is necessary, in each row and each column to reach 364 (or to exceed it).

This shows that Gordon has to get to 11 blocks south, 3 blocks east.

M3. At a horse show, a driver is demonstrating his ability to drive his pony and two-wheeled trap in a tight circle. The wheels of the trap are 5 ft apart and for the tightest circle, the outer wheel makes one and a quarter turns for every turn of the inner wheel. What is the circumference of the circle made by the outer wheels?

Solution

Let the radius of the circle made by the inner wheels be r feet so the circle made by the outer wheels has radius $(r + 5)$ feet.

The inner wheels travel $2\pi r$ feet in one complete circle while the outer wheels travel $2\pi(r+5)$ feet.

Therefore

$$2\pi(r + 5) = \frac{5}{4} 2\pi r \Rightarrow 4r + 20 = 5r.$$

So $r = 20$ and the circumference of the circle made by the outer wheels is 50π feet (which is approximately 157 feet).

M4. A desert island has a total of twenty shipwrecked people living on it. The leader suggests that the food provisions should be shared between them. There is a total of twenty portions of provisions and she proclaims that each man will be given three portions, each woman will be given two portions and each child half a portion.

How many men, women and children are shipwrecked on the island?

Solution

Let the number of men be m, the number of women be w and the number of children be c.

$$m + w + c = 20 \qquad (1)$$

$$3m + 2w + \tfrac{1}{2}c = 20 \qquad (2)$$

$$\Rightarrow 6m + 4w + c = 40$$

$$5m + 3w = 20$$

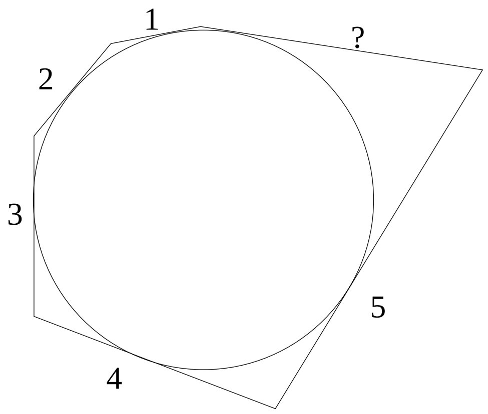
Thus possible pairs (m, w) are $(4,0)$, $(3,1\tfrac{2}{3})$, $(2,3\tfrac{1}{3})$, $(1,5)$, $(0,6\tfrac{2}{3})$.

But we cannot have fractions of a person and we know there is at least one woman so $w = 5$.

Thus $m = 1$ and this gives $c = 14$ from both (1) and (2).

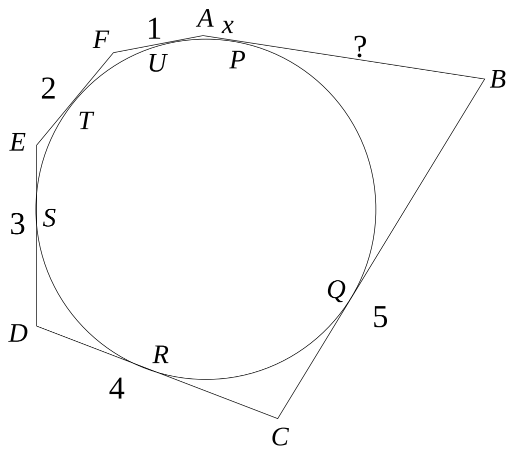
There is 1 man, 5 women and 14 children.

M5. In the hexagon shown, all edges are tangent to the circle. If their lengths are 1, 2, 3, 4 and 5 as illustrated, what is the length of the remaining edge?



Solution

From each vertex of the hexagon, the two tangents have the same length. Each edge is made up of two such tangent lines.



Introduce letters A, B, C, D, E, F for the hexagon and P, Q, R, S, T, U for the points of contact. Also, let $AP = x$.

So

$$\begin{aligned} AU = x &\Rightarrow UF = 1 - x = FT \\ &\Rightarrow TE = 1 + x = ES \\ &\Rightarrow SD = 2 - x = DR \\ &\Rightarrow RC = 2 + x = CQ \\ &\Rightarrow QB = 3 - x = BP \end{aligned}$$

Therefore

$$AB = AP + PB = x + (3 - x) = 3.$$

Thus the length of the remaining edge is 3.