2008 Middle Set 1 solutions

M1. On a 26-question test, 8 points were credited for each correct answer and 5 points were deducted for each wrong answer. If all questions were answered, how many were correct if the score was zero?

If Fred and Bernie both scored more than zero, but Fred scored 10 times as many points as Bernie, how many did Fred score correctly and how many did Bernie score correctly? (Again assume that all questions were answered).

Solution 1.

If x answers were correct and y were wrong, then $x + y = 26$ and $8x - 5y = 0$.

So $8x = 5(26 - x)$ i.e. $13x = 130$. So 10 questions are answered correctly for a score of zero.

Let Fred get x_1 correct and y_1 wrong and Bernie gets x_2 correct and y_2 wrong. We then have the following equations: $x_1 + y_1 = 26, x_2 + y_2 = 26, 8x_1 - 5y_1 = 10(8x_2 - 5y_2)$.

This last equation shows that y_1 must be even and so x_1 must be even. It also shows that x_1 must be divisible by 5. Hence x_1 is a multiple of 10. By the first part, it is bigger than 10 and on a 26-question test can only be 20. Thus Fred scored 20 questions correctly giving a points total of 130 while Bernie scored 11 correctly giving a score of 13.

Alternative Solution without algebra:

Each time one more question is answered correctly, the score changes by 13 marks. If initially none are correct, the score is $-5 \times 26 = -130$. So 10 questions answered correctly are required to make the score 0.

The number of questions over 10 which are answered correctly gives a score which is a multiple of 13. But if Fred scored 10 times as many points as Bernie, he must have scored 130 points and Bernie 13 since the total number of questions is 26. So Fred got 20 questions correct and Bernie got 11.

M2. Sam had never fully understood the points system in football and felt that the scoring of goals should be encouraged. His idea is that 10 points should be awarded for a win, 5 points for a draw and 1 point for each goal scored, whatever the result of the match. Therefore even if you are losing 0-5 and have no hope of winning, a goal scored might make all the difference between promotion and relegation. This was tried with three teams, Hubs, Dins and Rungs. Each team scored at least one goal in every match and no team played another more than once. Hubs scored 8 points, Dins 14 points and Rungs scored 9 points. Find the score in each match.

Solution

The total number of points awarded is 31. The number of points available per match here is 12. So only two games were played.

Dins never loses as Hubs and Rungs never win.

If Dins plays twice, Rungs plays once so must draw with Dins at a score of 4 – 4. But then since Dins scores at least one goal against Hubs, their total would the be at least 15. So it follows that Dins only plays once. This means that Hubs and Rungs play each other. Either Hubs or Rungs only plays once so the score between them must be 3 – 3 or 4 – 4. It cannot be 4 – 4 as this would give Hubs at least 9 points. So it must be 3 – 3 and Hubs plays only once. Thus Rings then plays Dins and loses 1 – 4.

M3. A dishonest market trader has doctored his balance scales so that one arm of the scales is longer than the other, but the pan A on the short side has been made heavier so that a balance is still achieved when the pans are empty. He is selling exotic mixed nuts at £1 for a 250g bag weighed on his scales. He is doing a roaring trade and has emptied seven 24kg sacks of the exotic nuts. This is partly due to his sales gimmick which is, for every 20 bags sold he gives one away "free, gratis and for nothing". The market regulator finds him out by putting a bag of nuts on pan A of the scales and finding that it balanced with 160g weight on pan B. The trader confesses but pleads that he is a good guy as he has been giving away nuts for free. The regulator agrees to take that into consideration in his calculations, but fines him six times the amount of money he made fraudulently by selling the exotic nuts. How much is the fine?

Solution

Let Wg be the actual weight of a bag of nuts. Then $Wx = 250$ where x is the scaling factor for the dishonest scales (actually the relative length of the arms of the scales). When reversed, we have $160x = W$. So $W^2 = 160 \times 250 = 40\,000$. Thus $W = 200$.

When he sells 20 bags, he makes £20. The free bag is also "sold", so the total weight sold is 21×200 grams $= 4200$ grams $= 16.8 \times 250$ grams. So the 21 bags should have been sold for £16.80. Thus he makes £3.20 fraudulently on every 4200 grams sold.

To find the total amount made, we find how many lots of 21 bags can be made out of seven 25kg sacks. Working in kilograms this is

$$\frac{7 \times 24}{4.2} = 40.$$

So the amount of the fine will be $40 \times £3.20 \times 6 = £768$.

M4. Three boys went to the local scout jumble sale and visited the book stall. Afterwards they compared their purchases. Harry said to Jake, " if I give you 6 of my football books for one of your scouting books, you would have twice as many books as I would have". Then Dai said to Harry, "but if I gave you 14 of my books about cars for one of your scouting books, then you would have three times as many books as I would have". Finally, Jake said to Dai, "if I gave you 4 of my Beano annuals for one of your scouting books, then you would have six times as many books as I would have". How many books did each boy have?

If they all had the same number of scouting books and the same number of Beano annuals, and only Harry had football books, how many books of each type did they have? (Assume that they did not have any other types of books.)

Solution

Let J, H, D be the number of books that Jake, Harry and Dai have respectively. We then have the following three equations that these numbers must satisfy.

$$J + 5 = 2(H - 5) \tag{1}$$

$$H + 13 = 3(D - 13) \tag{2}$$

$$D + 3 = 6(J - 3) \tag{3}$$

Substitute for D from (3) in (2) to get

$$H = 18J - 115.$$

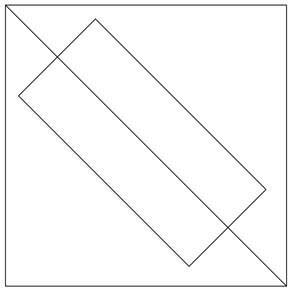
Now substitute for H in (1) to get $35J = 245$ so that $J = 7$. Hence $D = 21$ and $H = 11$.

Each boy had at least one scouting book. Harry had at least 6 football books. Dai had at least 14 books about cars. Jake had at least 4 Beano annuals.

Since they all had the same number of Beano annuals, then Harry must have at least 4 Beano annuals. But Harry has 11 books all together. So he must have 1 scouting book, 4 Beano annuals and 6 football books. Since Harry is the only one with football books, Dai must have 1 scouting book, 4 Beano annuals and 16 books about cars and Jake has 1 scouting book, 4 Beano annuals and 2 books about cars.

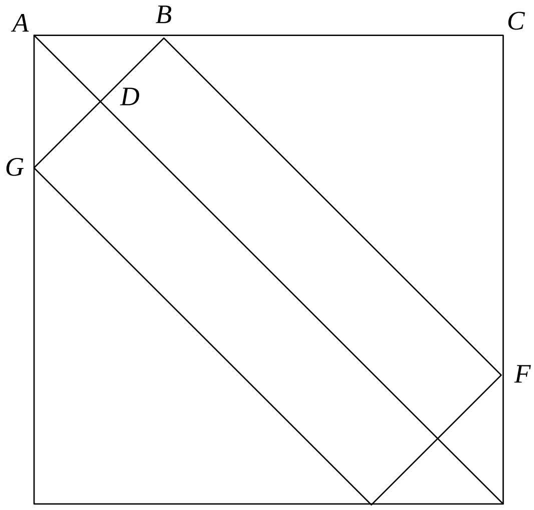
M5. Show that if a rectangle, which is twice as long as it is broad, can fit diagonally into a square as shown below, then it can also fit into the square with its sides parallel to the sides of the square.

Is this true if the rectangle is three times as long as it is broad? **Explain your answer**.



Solution

Consider the extreme case where the corners of the rectangle lie on the sides of the square.



Let the square have side length L. Let $BF = 2y$ and $BG = y$. So $BD = AD = \frac{1}{2}y$. So $AB^2 = AD^2 + BD^2 = \frac{1}{2}y^2$. So $AB = y/\sqrt{2}$.

Now $BC = CF$. So $2BC^2 = BF^2 = 4y^2$. So $BC = \sqrt{2}y$. So

$$L = AB + BC = \frac{y}{\sqrt{2}} + \sqrt{2}y = \frac{3}{\sqrt{2}}y.$$

So $BF = 2y = \frac{2\sqrt{2}}{3}L < L$ since $2\sqrt{2} < 3$.

So this extreme rectangle, and so any other rectangle which is twice as long as it is broad and which can fit in diagonally, could fit in with its sides parallel to the sides of the square.

Now suppose that $BF = 3y$. As before $AB = y/\sqrt{2}$ and now $2BC^2 = 9y^2$ so $BC = \frac{3}{\sqrt{2}}y$. So $L = \frac{4}{\sqrt{2}}y$. So $BF = 3y = \frac{3\sqrt{2}}{4}L$. Now $\frac{3\sqrt{2}}{4} > 1$ so this rectangle could *not* fit in with its sides parallel to the sides of the square.