2007 Middle Set 2 solutions

M1. In the village of Piffle, some of the animals are really strange. Ten percent of the dogs think they are cats and ten percent of the cats think they are dogs. All the other cats and dogs are perfectly normal. One day the official village animal psychologist tested all the cats and dogs in the village and found that twenty percent of them thought that they were cats. What percentage of them really were cats?

Solution

Suppose that there are n cats in the village and m dogs. Then $0.9n$ cats think they are cats and the other $0.1n$ cats think they are dogs. Also $0.9m$ dogs think they are dogs and $0.1m$ dogs think they are cats. So the total number of animals in the village that think they are cats is $0.9n + 0.1m$. Thus the fraction that think they are cats is

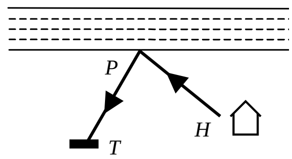
$$\frac{0.9n + 0.1m}{n + m} = 0.2$$

$$\Rightarrow 0.9n + 0.1m = 0.2n + 0.2m$$

$$\Rightarrow 0.7n = 0.1m \Rightarrow m = 7n.$$

But then actual fraction of cats is $\dfrac{n}{n + m} = \dfrac{n}{n + 7n} = \dfrac{1}{8}$. Thus the percentage is 12.5.

M2. Every morning, Ruth goes out to take water to her pet armadillo. She picks up her water bucket by her house at the point H on the diagram below, goes down to the edge of the straight river, fills the bucket with water and takes it to the armadillo's trough at T. She has been doing this so often that she knows exactly where the point P on the river bank is, so that she walks the shortest distance. Explain how you should choose the point P on the river bank so that the distance from H to T via P is shortest.



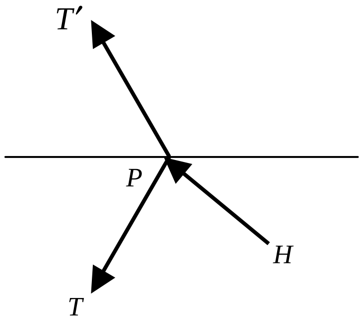
Solution

Regard the edge of the riverbank as a mirror and let T' be the mirror image of T. Then the distance from H to T via P is the same as the distance from H to T' via P.

But the shortest distance between any two points is a straight line (on a flat plane).

So the line from H through P to T' should be a straight line.

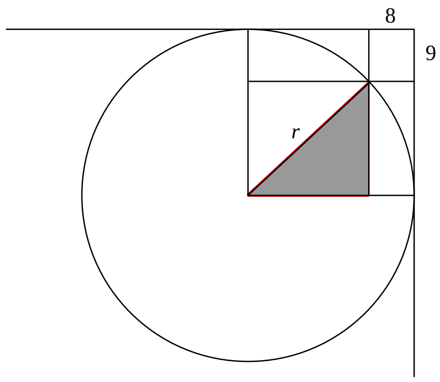
So the angle that HP makes with the riverbank will be the same as the angle $T'P$ makes with the riverbank, which, in turn, is the same as the angle TP makes with the riverbank.



The point on the riverbank should be chosen so that the angle that the line HP makes with the bank is the same as the angle that the line TP makes with the riverbank.

M3. A boy accidentally made an inky mark on the rim of an antique circular table. He tried to hide it by pushing the table into the corner of a rectangular room so that it was touching both walls. When his father came home, he discovered the mark but realised it could be cleaned off. However, pointing to the mark on the far side of the table, he set his son the following problem: he said “That mark is exactly 8 inches from one wall and 9 inches from the other wall. Without measuring it, what is the diameter of the table?”

Solution



Let the radius of the table be r inches, then the right-angled triangle shown, with one vertex at the centre, has sides of lengths r, $r - 8$, $r - 9$ inches.

So by Pythagoras' theorem, we have

$$r^2 = (r - 8)^2 + (r - 9)^2 .$$

Which gives

$$r^2 - 34r + 145 = 0$$

$$\text{i.e. } (r - 5)(r - 29) = 0.$$

Since r is at least 9 this shows that $r = 29$ and the diameter is 58 inches.

M4. In a far off country, Tanika's brother gets a weekly allowance of \$13. Being mathematicians, her parents use some special dice to work out how much Tanika should get each week. The dice are fair (i.e. unbiased) and each has six faces with a number on it. The numbers on each die are: 1, 5, 5, 10, 10 and 20. Tanika's allowance is determined by rolling two such dice and taking the sum, in dollars, of the numbers rolled. On average, will she get more or less than her brother and by how much? **Explain your answer.**

Solution

The table shows the possible amounts.

+	1	5	5	10	10	20
1	**2**	**6**	**6**	**11**	**11**	**21**
5	**6**	**10**	**10**	**15**	**15**	**25**
5	**6**	**10**	**10**	**15**	**15**	**25**
10	**11**	**15**	**15**	**20**	**20**	**30**
10	**11**	**15**	**15**	**20**	**20**	**30**
20	**21**	**25**	**25**	**30**	**30**	**40**

Each of the amounts (in bold) is equally likely and their total is

$$2 + 4 \times 6 + 4 \times 10 + 4 \times 11 + 8 \times 15 + 4 \times 20 + 2 \times 21 + 4 \times 25 + 4 \times 30 + 40 = 612.$$

So the average amount is $612 \div 36 = 17$.

Which means that, on average, Tanika will get \$4 per week more than her brother.

M5. A cyclist and a runner start off simultaneously around a racetrack each going at a constant speed in the same direction. The cyclist completes one lap and then catches up with the runner. Instantly the cyclist turns around and heads back at the same speed to the starting point where he meets the runner who has just finished his first lap. Find the ratio of their speeds.

Solution

Let the speeds of the cyclist and the runner be u and v respectively. Let the fraction of the circuit covered by the runner when he meets the cyclist for the first time be x. At this time the cyclist will have covered $1 + x$ circuits. Equating the times gives:

$$\frac{x}{v} = \frac{1 + x}{u} \tag{1}$$

$$\text{thus } ux = v(1 + x) \Rightarrow \frac{u}{v} = \frac{1 + x}{x}.$$

Similarly the time taken for the second phase:

$$\frac{1 - x}{v} = \frac{x}{u} \tag{2}$$

$$\text{giving } u(1 - x) = vx \Rightarrow \frac{u}{v} = \frac{x}{1 - x}.$$

From the two equations, the ratio of u to v gives

$$\frac{1 + x}{x} = \frac{x}{1 - x}$$

Hence

$$x^2 = 1 - x^2.$$

From this

$$x = \sqrt{\frac{1}{2}}$$

and this gives the ratio of the speeds as

$$\frac{u}{v} = \frac{1 + 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} + 1.$$