2007 Middle Set 1 solutions

M1. Find all the six-digit numbers using each of the digits 1, 2, 3, 4, 5, 6 such that the numbers formed

by the first digit is divisible by 1,
by the first two digits is divisible by 2,
by the first three digits is divisible by 3,
by the first four digits is divisible by 4,
by the first five digits is divisible by 5,
by the first six digits is divisible by 6.

Explain your reasoning.

Solution

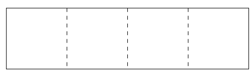
Denote the number by $abcdef$. Since $abcde$ is divisible by 5, then we must have $e = 5$. Since ab, $abcd$, $abcdef$ are all divisible by 2, then b, d, f must all be even numbers. Thus a and c take the values 1 or 3. Now if a number is divisible by 3, the sum of its digits is divisible by 3. Since b is either 2, 4 or 6 the sum of the first three digits is either 6, 8 or 10 respectively. But only the first of these is divisible by 3.

Thus b must be 2 and the first three digits are 123 or 321. Now d is either 4 or 6 and we can check which of the numbers 1234, 3214, 1236, 3216 are divisible by 4.

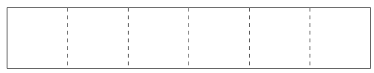
The possibilities are 1236 or 3216. Thus the whole number will be 123654 or 321654 and both of these are divisible by 6. Thus these are the only two numbers which satisfy all the divisibility conditions.

M2. In this question, you are only allowed to shade complete squares.

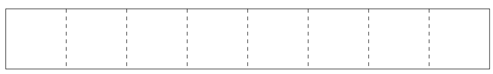
(a) In how many different ways is it possible to shade one half of this rectangle?



(b) In how many different ways is it possible to shade one third of this rectangle?



(c) In how many different ways is it possible to shade one quarter of this rectangle?

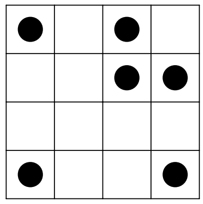


(d) In how many different ways is it possible to shade one fortieth of a rectangle made up of 80 squares?

Solution

(a) 6 ways
(b) 15 ways
(c) 28 ways
(d) $\frac{1}{40}$ is still 2 squares. The first square can be any of 80 and the second any of 79. Multiplying 80 and 79 gives 6320 but this counts each pair twice. So there are 3160 ways.

M3. The diagram shows a 4 × 4 grid containing 6 black spots. These 6 spots are so placed that no three of them lie in a line, either horizontally, vertically or diagonally, but if you add one more spot, there will always be such a line of three spots.



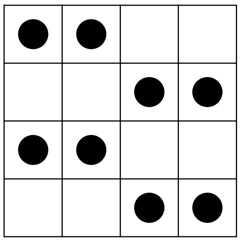
What is the largest number of black spots you can place on such a 4 × 4 grid with this property i.e. no three spots are in a line but if you add any one spot there will always be a line of three?

What is the smallest number of black spots you can place on such a 4 × 4 grid with this property?

Explain your answers.

Solution

The largest number is 8 and these can be placed as shown below, but there are several other possibilities. If you have 9 spots or more there must be at least 3 in one row, so 8 is the maximum.



The smallest number is 4 with the spots placed either at the four corners or in the middle four squares.

If 3 spots were enough, two spots would have to lie in one row and the third in a different row. Now consider a row with no spots in it. The only squares in that row which will give 3 in a line are those which lie on one of the two lines from the two spots in one row to the one spot in the other row. So a black spot could be placed in either of the two remaining squares in that row and that spot would not lie in a line of 3.

OR

If 3 spots were enough then two would have to lie in one row with the third in a different row. But the same applies to columns. So the three would have to lie in squares which form three of the vertices of a rectangle of squares.

M4. Calum was floating down river on a raft, when, half a mile downstream, his brother Duncan set off in a canoe. Duncan paddled downstream as quickly as he could, then turned round and paddled back again, still at his best pace. He and arrived back at his starting point just as Calum floated by.

Assuming Duncan's best pace in still water is 10 times that of the river current, how far did he paddle?

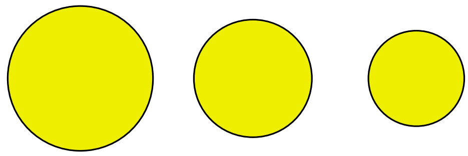
Solution

Suppose that the river current is x mph. So the time for Calum to reach Duncan's starting point is $1/(2x)$ hours. When Duncan paddles downstream he will travel at a rate of $11x$ mph and when he paddles upstream he travels at $9x$ mph. Suppose that he paddles d miles downstream. That will take him $d/(11x)$ hours. Paddling back up again will take him $d/(9x)$ hours. Since the total time taken is the same as it takes Calum to get to Duncan's starting point we have the equation:

$$\frac{d}{11x} + \frac{d}{9x} = \frac{1}{2x} \Rightarrow d\left(\frac{1}{11} + \frac{1}{9}\right) = \frac{1}{2} \Rightarrow d\frac{20}{99} = \frac{1}{2} \Rightarrow d = \frac{99}{40}.$$

Thus the total distance travelled up and down is twice that i.e. 99 / 20 miles.

M5. Three large pancakes, as shown, each of the same thickness are to be shared equally among four people. The diameters of the pancakes form a Pythagorean triple. Without measuring, show how to cut the pancakes to make a total of just five pieces so that each person can get the same amount.
Explain your reasoning.



Solution

If the diameters are a, b, c then $a^2 = b^2 + c^2$. Since the pancakes are all of the same thickness, the amount of pancake is proportional to its area. The areas of the pancakes are $\frac{1}{4}\pi a^2$, $\frac{1}{4}\pi b^2$, $\frac{1}{4}\pi c^2$ so the total area is $\frac{1}{2}\pi a^2$. So cutting the largest pancake in half gives two of the four equal portions. Now place the smallest pancake over the middle one and cut off half of the portion between the two pancakes. That has area $\frac{1}{2}\left(\frac{1}{4}\pi b^2 - \frac{1}{4}\pi c^2\right)$. That part together with smallest circle then has total area $\frac{1}{2}\left(\frac{1}{4}\pi b^2 + \frac{1}{4}\pi c^2\right) = \frac{1}{8}\pi a^2$. Also the part that remains on the middle circle has the same area so we get four equal portions.