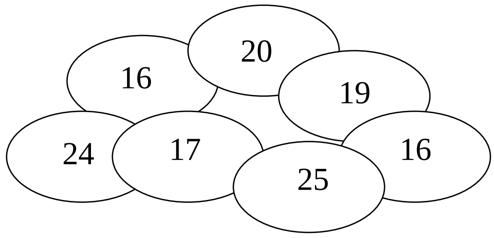
2006 Middle Set 2 solutions

M1. A merchant has an odd collection of barrels of wine and one barrel of beer as shown with their capacities in gallons.

He gets rid of all of it by selling off some barrels of wine to one customer, twice that quantity of wine to another customer and keeping the barrel of beer for himself. How many gallons did the barrel of beer contain?



Solution

Since the wine is divided so that one customer gets twice as much as the other, the total number of gallons of wine is a multiple of 3.

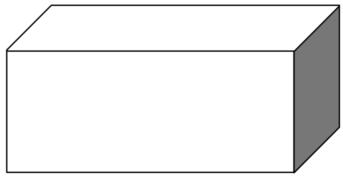
Now the total number of gallons altogether is 16+20+19+24+17+25+16=137. So we need to remove one barrel and be left with a multiple of 3. Since 137 leaves a remainder 2 on division by 3, the possible ones to remove are 20 and 17.

If we remove 20, we get $117 = 3 \times 39$. So one customer would get 39 gallons of wine. But no combination of the remaining barrels of wine gives 39.

If we remove 17 we get $120 = 3 \times 40$. So one customer gets 40 = 16 + 24 gallons of wine and the barrel of beer contains 17 gallons.

M2. For Helen's birthday, she was given a box of chocolates. The area of the top of the box was 338 cm², the area of the side was 104 cm² and the area of the end was 52 cm².

What was the volume of the box?



Solution

$lb = 338$ $\qquad$ $bh = 52$ $\qquad$ $lh = 104$

$l = \dfrac{338}{b}$ $\qquad$ $b = \dfrac{52}{h}$ $\qquad$ $h = \dfrac{104}{l}$

$l = 338 \times \dfrac{h}{52}$

$l = \dfrac{338}{52} \times \dfrac{104}{l}$

$l^2 = 676$ $\qquad$ $l = 26$ $\qquad$ $b = \frac{52}{4} = 13$ $\qquad$ $h = \frac{104}{26} = 4$

Volume = $26 \times 13 \times 4 = \mathbf{1352}$ cm^3.

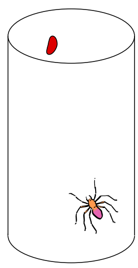
OR

$lb = 338$ $\qquad$ $bh = 52$ $\qquad$ $lh = 104$

Multiplying these gives $(lb)(bh)(lh) = 338 \times 52 \times 104$ which is $(lbh)^2$ i.e. V^2.

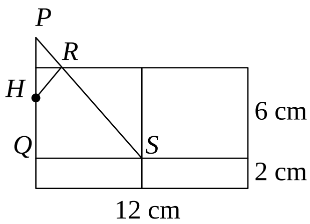
Hence $V^2 = 338 \times 52 \times (2 \times 52) = 676 \times 52^2 = 24^2 \times 52^2$. Thus $V = 24 \times 52 = 1352$ cm^3.

M3. A cylindrical beaker 8 cm high and 12 cm in circumference was standing on a table. On the *inside* of the beaker, 2 cm from the top, is a drop of honey. Diametrically opposite the honey and lower down is a spider which is on the *outside* of the beaker, 2 cm from the bottom. What is the shortest distance the spider has to walk to reach the honey?



Solution

The diagram shows the curved surface unrolled.



To reach the honey, the spider should crawl in a straight line from P, the image of H, to S.

$$\begin{aligned} SP^2 &= PQ^2 + SQ^2 \\ &= 8^2 + 6^2 \\ &= 100 \end{aligned}$$

Thus the spider must crawl 10 cm to reach the honey.

M4. An eccentric and wealthy man made a will in which he left £55,000 each year to secondary schools and primary schools in the area where he lived. The secondary schools were to receive £3,500 each and the primary schools £2,000 each. The terms of the legacy were that all the money had to be spent each year. But the eccentric clause was that, in each year, the number of secondary schools which benefit must be different from the number of secondary schools which had benefitted in all preceding years and the same must hold true for the number of primary schools. For how many years could the legacy be spent?

Solution

Suppose in one year, s secondary schools benefited and p primary schools benefited. So $3500s + 2000p = 55000$ i.e. $7s + 4p = 110$. Suppose the maximum number of secondary schools benefit and the minimum number of primary schools. So we want the smallest positive integer p such that $110 - 4p$ is divisible by 7. This is $p = 3$ which gives $s = 14$. To get other solutions, increase p by multiples of 7 and stop when the corresponding value of s becomes negative. Thus

p	3	10	17	24
s	14	10	6	2

So the legacy runs for 4 years.

M5. Daryl put some black stones and some white stones into a bag. He then asked Ran to reach into the bag, without looking in, and draw out a stone. Ran drew out a black stone. Daryl asked Ran to draw out another stone and, once again, Ran drew out a black stone.

"There must be more black than white stones in the bag ," said Ran. "I wonder what the probability is of my drawing a black stone on a third try?"

Daryl replied," Exactly nine tenths of what it was of drawing a black stone on your first draw."

Daryl told Ran that he had put "ten, give or take two or three" stones into the bag. How many stones were in the bag at the start?

Solution

Suppose that there are n stones in the bag, of which b are black and w are white.

The probability of drawing a black stone on the first draw is $\frac{b}{b+w}$ and after drawing 2 black balls it is $\frac{b-2}{b+w-2}$. Hence,

$$\frac{b-2}{b+w-2} = \frac{9}{10}\frac{b}{b+w}.$$

This gives $10b^2 + 10bw - 20b - 20w = 9b^2 + 9bw - 18b$,

We know that $w = n - b$ and n is between 7 and 13; substituting and simplifying gives $b = \frac{20n}{n+18}$.

The only value of n that makes b an integer for n between 7 and 13 is, where $n = 12$, where $b = 8$. This makes $w = 4$.