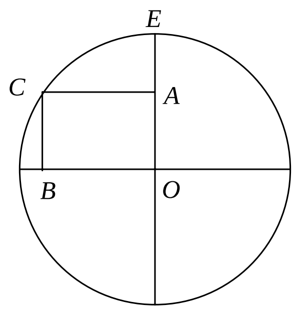
2006 Middle Set 1 solutions

M1. Lord Snooty's park is in the shape of a perfect circle of diameter 5 miles. There are two monuments in the park. To reach the first – commemorating the Battle with the Bash Street Kids – enter the park at its most northerly point and proceed due south for a mile. To reach the second monument – erected on the occasion of the 50^{th} Birthday of Dennis the Menace – from the first monument, go due west until the perimeter of the park is reached then turn due south and proceed for one and a half miles.

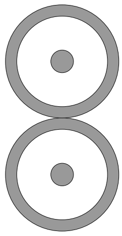
How far apart are the two monuments (as the crow flies) ?

Solution

The first monument is at *A* and the second at *B*. Note that to reach the second monument you proceed due south for $1 + 1\frac{1}{2} = 2\frac{1}{2}$ miles which is the same as the radius of the circle. So the second monument lies on the east-west diameter. Note that *A* and *B* are at opposite ends of the diagonal of a rectangle. The other diagonal, *OC*, joins the centre to a point on the circumference and so has length 2½ miles. Since the diagonals of a rectangle have equal length this is the distance between the two monuments.



M2. The diagram shows two wheels of the same diameter. The lower wheel is fixed and the upper wheel rotates without slipping about the lower wheel, the two wheels always being in contact. How many times does the upper wheel turn on its axis in making a complete revolution of the lower wheel?



Solution

Make a mark at the top of the upper wheel. On the journey round the lower wheel, this mark will be in contact with the lower wheel at the lowest point of the lower wheel when half the journey has been made. At this stage, the mark on the moving wheel is again at the top of it and so the moving wheel has made one complete revolution about its own axis. Thus there will be two complete revolutions for the whole journey.

M3. Two young mountaineers were descending a mountain quickly at 6 miles per hour. They had left the hostel late in the day, had climbed to the top of the mountain and were returning by the same route. One said to the other "It was three o'clock when we left the hostel. I am not sure if we will be back before nine o'clock." His companion replied "Our pace on the level was 4 miles per hour and we climbed at 3 miles per hour. We will just make it." What is the total distance they would cover from leaving the hostel to getting back there?

Solution

Let x be the number of miles covered ascending and y the number of miles on the level on the outward journey. So time taken to the summit is $x/3 + y/4$ hours. The time taken to get back is $x/6 + y/4$ hours. So the total time taken is the sum of these two which is $(x + y)/2$. So $x + y = 12$ and the total distance was 24 miles.

M4. In a snowball 'fight', where snowballs are identical spheres, your opponents have stacked their snowballs in a square pyramid. You are about to count the snowballs along the bottom edge of the opponent's stack when one appears with another snowball. After giving him a telling off, the opposition's leader takes apart the square pyramid and builds a new, triangular pyramid using all the original snowballs and the extra one. Find two possible values for the number of snowballs that your opponents now have.

Solution

The total in the square pyramid is $1^2 + 2^2 + 3^2 + \ldots + n^2 = S$.
By a known result $S = \frac{1}{6}n(n+1)(2n+1)$.

The total in the triangular pyramid is $\dfrac{1\times 2}{2} + \dfrac{2\times 3}{2} + \dfrac{3\times 4}{2} + \ldots + \dfrac{m(m+1)}{2} = T$.

Hence

$$\begin{aligned}
2T &= 1\times(1+1) + 2\times(2+1) + 3\times(3+1) + \ldots + m(m+1) \\
&= \left(1^2 + 2^2 + 3^2 + \ldots + m^2\right) + \left(1 + 2 + 3 + \ldots + m\right) \\
&= \frac{1}{6}m(m+1)(2m+1) + \frac{1}{2}m(m+1) \\
&= \frac{1}{6}m(m+1)\left[2m+1+3\right] = \frac{1}{3}m(m+1)(m+2)
\end{aligned}$$

i.e. $$T = \frac{1}{6}m(m+1)(m+2)$$

So for some values of n, m we require that $S + 1 = T$.
This holds for $n = 5$, $m = 6$ in which case the number of snowballs is 56. It also holds for $n = 9$, $m = 11$ in which case the number of snowballs is 286.
[In general it needs a solution to $n(n+1)(2n+1) + 6 = m(m+1)(m+2)$.
That may well have infinitely many solutions.]

Tabulating might be more accessible:

	1	2	3	4	5	6	7	8	9	10	11	
S	1	5	14	30	55	91	140	204	285	…		
T	1	4	10	20	35	56	84	120	165	220	286	…

M5. Three cyclists are out for the day. Two are on a tandem and one on an ordinary cycle. Disaster struck when the ordinary cycle was stolen while they were having lunch in a café. They were left with the tandem and 20 miles to go. The tandem has to have two riders and the third person walks. Anne can walk a mile in 20 minutes, Sam in 30 minutes and Oscar in 40 minutes. The tandem travels at 20 miles per hour no matter which pair is riding it. What is the shortest time for all three to get home?

Solution

The method is that the tandem and a walker start off together. The tandem goes a certain distance, stops and one gets off and starts to walk. The other stays with the tandem to wait for the walker to catch up. Then the two get on and cycle after the walker who was originally on the tandem. When they catch up with this walker, the whole method starts again. Since Oscar is the slowest walker, he should always be on the bike. So either Anne or Sam start walking. For shortest time, the change over should be made so that the tandem and the second walker arrive home together.

So let tandem cycle for n minutes and so will cover $n/3$ miles. If Anne starts walking first, her total time to get home will be $\frac{\frac{n}{3}}{\frac{1}{20}} + \frac{\left(20 - \frac{n}{3}\right)}{\frac{1}{3}} = 60 + \frac{17n}{3}$. The total time for Sam will be $n + \frac{\left(20 - \frac{n}{3}\right)}{\frac{1}{30}} = 600 - 9n$. So we require that $600 - 9n = 60 + \frac{17n}{3}$. This gives that $n = \frac{3 \times 540}{44}$. So the total journey time works out to be 268.6 minutes i.e. 4 hours and 28.6 minutes.