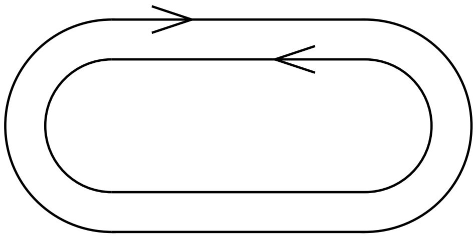
Junior Division 2017-2018 Round 2 Solutions

J1.



Trains on the Glasgow Subway depart every 4 minutes, and a complete circuit takes 24 minutes. Ewan sets off at 8.30 am on a train round in one direction at the same time as another train leaves in the opposite direction. How many trains will he pass on a complete circuit back to his starting station? (Do not count trains at the start or end station.)

Solution 1

At the start of his journey Ewan will see the 8.06 in the station, having just completed its journey in the other direction.

At the end of his journey at 8.54 he will see the 8.54 in the station, just about to leave in the other direction.

At some point around the circuit he will pass all the trains leaving between these, i.e. those leaving at 8.10, ... 8.50.

So he sees 11 trains.

Solution 2

Assume that the trains travel at constant speed around the circuit.

Then at 8.30 there will be a train 4 minutes away, which Ewan will pass after 2 minutes, as they travel towards each other. As they pass there will be another train approaching 4 minutes away, which Ewan will pass after another 2 minutes. So Ewan will pass trains every 2 minutes i.e. at 8.32, 8.34, ... 8.52 until he meets another train in the station at 8.54.

So he passes 11 trains on his journey.

{*In fact the trains will not travel at constant speed - there will be station stops around the circuit. But still Ewan will have to pass all the trains as if they were travelling at constant speed, so the answer is still 11.*}

J2. Some people think it is unlucky if the 13th day of a month falls on a Friday. Show that in every calendar year (non-leap or leap) there will always be at least one such unlucky Friday but that there can be no more than three.

Solution 1

Taking 2018 as an example non-leap year, the table shows the days that the 13th falls on

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sa	Tu	Tu	F	Su	W	F	M	Th	Sa	Tu	Th

Each day of the week occurs in this list at least once. So whichever day of the week 13th Jan falls on in a non-leap year there will always be at least one Friday 13th in the year.

Similarly no day of the week appears more than 3 times in this list. So whichever day of the week 13th Jan falls on in a non-leap year there can be no more than 3 Friday 13ths in the year.

For a leap year, the table becomes

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sa	Tu	W	Sa	M	Th	Sa	Tu	F	Su	W	F

As above, each day of the week occurs so there is at least one Friday 13th in the year.

Also no day of the week occurs more than 3 times so there are at most 3 Friday 13ths in the year.

J2. *Solution* 2

Suppose that the 13th of two months are Fridays. Then the later Friday must be n days after the first, where n is a multiple of 7.

Consider the case of a non-leap year. Then the 13th of February is 31 days after the 13th of January, the 13th of March is 28 days after the 13th of February, and so on. We can display this as

Jan		Feb		Mar		Apr		May		June		July		Aug		Sep		Oct		Nov		Dec
	31		28		31		30		31		30		31		31		30		31		30	

Since we are interested in gaps that sum to a multiple of 7, we can replace 31 by 3 and replace 30 by 2 as they as 3 more than a multiple than 7 and 2 more, respectively. Also, 28 is a multiple of 7 and is replaced by 0. This leads to the table of reduced gaps

Jan		Feb		Mar		Apr		May		June		July		Aug		Sep		Oct		Nov		Dec	
	3		0		3		2		3		2		3		3		2		3		2		(A)

Thus the day of the week of 13th Feb and also 13th Mar will be 3 on from that of 13th Jan; then the day of the week of 13th April will be 3 + 0 + 3 = 6 days on from the day of the week of 13th Jan; and so on. This leads to the table of cumulative sums of gaps in (A) as

Jan		Feb		Mar		Apr		May		June		July		Aug		Sep		Oct		Nov		Dec	
	3		3		6		8		11		13		16		19		21		24		26		(B)

Again we can replace with remainders after division by 7, giving

Jan		Feb		Mar		Apr		May		June		July		Aug		Sep		Oct		Nov		Dec	
	3		3		6		1		4		6		2		5		0		3		5		(C)

Now every possible remainder when dividing by 7 (i.e. 0, 1, 2, …, 5, 6) appears in (C), which means that, whatever day of the week 13th Jan falls on, the 13th of some later month will be a Friday (for instance, if 13th Jan is a Wednesday, then 13th August will be a Friday because Fridays occur 2 days after Wednesdays and 2 appears between July and Aug in (C)). Thus there will always be at least one unlucky Friday in every non-leap year.

In the case of a leap year, there are 29 days in February and so, in the comparable tables, the gaps from Feb on will be increased by one. In particular, the table of cumulative sums of gaps for (B) will now end with 27 between Nov and Dec and the table corresponding to (C) will become

Jan		Feb		Mar		Apr		May		June		July		Aug		Sep		Oct		Nov		Dec	
	3		4		0		2		5		0		3		6		1		4		6		(D)

This list also contains all possible remainders when dividing by 7 and so, as before, there must be at least one unlucky Friday in every leap year.

Notice that 3 appears exactly three times in (C). So, if 13th Jan is a Tuesday, then 13th of Feb, Mar and Nov will Fridays and there will be three unlucky Fridays in that year. Likewise, 0 appears exactly twice in (D), meaning that, if 13th Jan is a Friday, then 13th Apr and 13th July will be Fridays and there will be three unlucky Fridays in that year. Since no non-zero integer appears more than three times and 0 appears no more than twice in (C) or (D), there can never be more than three unlucky Fridays in any year, non-leap or leap.

J3. Three energy saving improvements are advertised to save 25%, 55% and 20% of the energy used. A homeowner makes these three improvements in succession. What overall percentage saving can be expected?

Solution

Proportion of original energy use after improvements

$$= (1 - 0.25)(1 - 0.55)(1 - 0.2)$$

$$= 0.75 \times 0.45 \times 0.8$$

$$= 0.27$$

Saving 73%.

J4. A strange announcement was made on the radio about a local election with three candidates: Mrs Allan, Mr Baxter and Ms Campbell.

"Mrs Allan beat Mr Baxter by one eighth of the total votes cast.

Mr Baxter beat Ms Campbell by a seventh of the total votes cast.

The votes cast for Mrs Allan was 350 fewer than 3 times Ms Campbell's votes."

How many votes did each candidate get?

Solution

Let a, b, c be the number of votes for Mrs Allan, Mr Baxter and Ms Campbell respectively.

$$a = b + \tfrac{1}{8}(a + b + c) \qquad b = c + \tfrac{1}{7}(a + b + c) \qquad a = 3c - 350 \ldots (3)$$

$$8a = 8b + a + b + c \qquad 7b = 7c + a + b + c$$

$$7a = 9b + c \ldots (1) \qquad 6b = 8c + a \ldots (2)$$

Substitute (2) into (1)

$$7a = 12c + \tfrac{3}{2}a + c$$

$$11a = 26c$$

$$a = \frac{26}{11}c$$

Substitute into (3)

$$\frac{26}{11}c = 3c - 350$$

$$26c = 33c - 3850$$

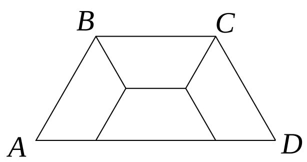
$$7c = 3850$$

$$c = 550$$

$$\text{So,} \quad a = 1300 \quad \text{and} \quad b = 950$$

Mrs Allan got 1300 votes, Mr Baxter got 950 votes and Ms Campbell got 550 votes.

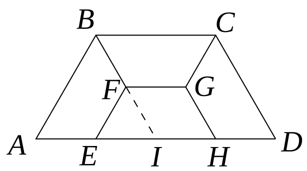
J5. A trapezium *ABCD* is split into four identical trapezia as shown below.



Given that *AB* has length 6 cm, find the area of *ABCD.*

Solution

We insert extra labels and line as shown below.



The trapezia are identical so $AB = BC = CD = EH$ and each is 6 cm.

All of the short sides of the trapezia are equal i.e. $EF = FG = GH = AE = BF = GC = DH$.

Since $FGHI$ is a parallelogram (actually a rhombus) $FI = GH = BF$ and as $BCDI$ is also a parallelogram $BI = CD$ hence $BF = \frac{1}{2}BI = 3$ cm as do all the other short lengths.

So $AD = 3 + 6 + 3 = 12$ cm

Consider triangle ABE: $BE^2 = 6^2 - 3^2 = 36 - 9 = 27$.

The area of a trapezium is half of the sum of the parallel sides multiplied by the distance between them.

Area of $ABCD = \frac{1}{2}(12 + 6) \times 3\sqrt{3} = 27\sqrt{3}$ cm^2 .