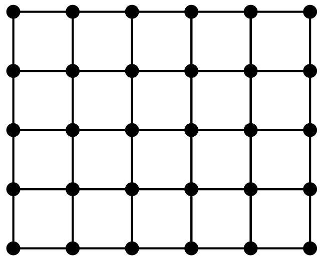
2015-2016 Junior Solutions Round 2

J1. The diagram represents a rectangular net. The net is made from string knotted together at the points shown. The strings are cut a number of times; each cut severs precisely one section of string between two adjacent knots. What is the largest number of such cuts that can be made without splitting the net into two separate pieces?



Solution

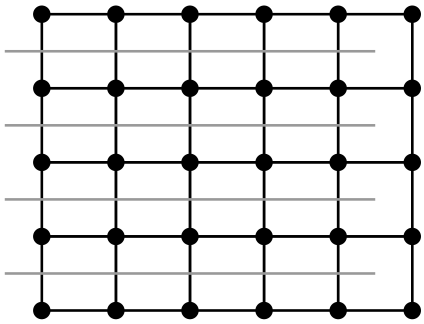
There are $6 \times 5 = 30$ knots.

To connect these 30 knots requires a minimum of 29 strings.

There are $5 \times 5 = 25$ horizontal strings, and $6 \times 4 = 24$ vertical strings, 49 strings in all.

So at most $49 - 29 = 20$ strings can be cut.

This is possible:



J2. In a diving competition, five judges each award a whole-number score from 1 to 10 and an average mark is then calculated. However there are three different ways of measuring the average: mean, mode and median. After a particular set of scores were given, an argument arose as to which measure should be used, as this would lead to three different final marks being awarded: 7, 8 or 9. Work out all the different possible scores that could have been awarded. Which mark would match with each measure?

Solutions

If the **mean** is 9, the scores could be 9, 9, 9, 9, 9 or 8, 9, 9, 9, 10 or 8, 8, 9, 10, 10 or 7, 9, 9, 10, 10. But then the median would also be 9, whereas it must be 7 or 8.

So the mean is **not** 9.

If the **median** is 9, the three highest scores must be

9, 9, 9 making the mode 9 also
or 9, 9, 10 making the mode 9 also (or undefined)
or 9, 10, 10 making the mode 10 (or undefined).

So the median is **not** 9.

Hence the **mode** must be 9.

If there are three or more 9s the median must be 9 also.
If the highest numbers are 9, 9, 10 the median must be 9 also.
So the highest numbers must be two 9s with no other repeats.

If the median is 8 then the three highest numbers are 8, 9, 9.

To make the mean 7, the other two numbers must add to $7 \times 5 - 8 - 9 - 9 = 9$.
So the possibilities are 4, 5 or 3, 6 or 2, 7 (but not 1, 8 as the mode would be undefined).

If the median is 7, then the three highest numbers are 7, 9, 9.
To make the mean 8 the other two numbers must add to $8 \times 5 - 7 - 9 - 9 = 15$.
But no two numbers less than or equal to 7 add to 15.

So there are exactly three valid score combinations which are:

	Mean	Median	Mode
4, 5, 8, 9, 9	7	8	9
3, 6, 8, 9, 9	7	8	9
2, 7, 8, 9, 9	7	8	9

In each case the mean is 7; the median is 8; and, the mode is 9.

J3. An artist who lived in the Outer Hebrides wished to replenish his larder and paid a visit to the local farmer.

"You have some fine birds there", he said, "what do they weigh?"

"Well sir", said the farmer, "a turkey and a duck together weigh twice as much as a goose. A goose and a chicken together weigh twice as much as a duck. A goose, a duck and a chicken together weigh twice as much as a turkey. And a duck weighs 8½ pounds."

The artist thanked him and went home to work out the weight of the others.

Find the weight of each of the birds.

Solutions

Let the weights of a turkey, duck, goose and chicken be t, d, g and c pounds respectively.

$$t + d = 2g \ldots (1) \qquad g + c = 2d \ldots (2) \qquad g + d + c = 2t \ldots (3) \qquad d = 8\tfrac{1}{2}$$

Using (2) and (3)

$$\begin{aligned} 3d &= 2t \\ 2t &= 3 \times 8\tfrac{1}{2} \\ t &= 12\tfrac{3}{4} \end{aligned}$$

Using (1)

$$\begin{aligned} 12\tfrac{3}{4} + 8\tfrac{1}{2} &= 2g \\ 2g &= 21\tfrac{1}{4} \\ g &= 10\tfrac{5}{8} \end{aligned}$$

Using (2)

$$\begin{aligned} c + 10\tfrac{5}{8} &= 17 \\ c &= 6\tfrac{3}{8} \end{aligned}$$

Answer: A duck weighs $8\tfrac{1}{2}$ lb, a turkey weighs $12\tfrac{3}{4}$ lb, a goose weighs $10\tfrac{5}{8}$ lb and a chicken weighs $6\tfrac{3}{8}$ lb.

J4 A pairs jousting tournament in which each knight would fight every other knight in the competition (unless he had to withdraw due to serious injury) was just about to start. Some unknown knights rode up and asked to be allowed to take part in the tournament. It was decided to include the unknown knights and 26 more pairs competitions had to be scheduled.

How many knights were taking part originally and how many unknown knights arrived?

Solutions

Suppose that there were s jousters at the start and an extra y knights rode up. Then $\frac{1}{2}s(s-1)$ jousts were scheduled originally.

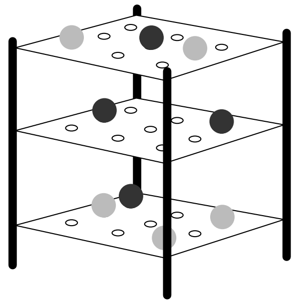
The new number of jousts required $= \frac{1}{2}(s+y)(s+y-1)$.

$$\frac{1}{2}(s+y)(s+y-1) - \frac{1}{2}s(s-1) = 26 \Rightarrow y(2s+y-1) = 52.$$

Now, y is a whole number and also a factor of 52; i.e. 1, 2, 4, 13, 26 or 52.
"Some" knights rode up so y is not 1.

If $y = 13$, 26, or 52, s is negative, so y is either 2 or 4.
If $y = 2$, then s is not a whole number; if $y = 4$, $s = 5$.
So originally there were 5 knights and another 4 appeared.

J5 Imagine a three-dimensional version of noughts and crosses: two players take it in turn to place different coloured marbles in a $3 \times 3 \times 3$ cube arrangement as shown in the diagram.



The object of the game is to create as many lines of three marbles of your own colour as possible.

How many different possible lines are there?

Solution 1

There are 8 lines within each of 3 layers of the large cube − total $8 \times 3 = 24$ lines.

Considering one vertical direction only, there are 5 lines not yet counted in each of 3 slices − $5 \times 3 = 15$ in total.

In the other vertical direction, there are 2 lines not yet counted in each of 3 slices - $2 \times 3 = 6$ in total.

Finally there are 4 lines between opposite corners of the large cube.

Hence the total number of lines is 49.

Solution 2

There are 7 lines through each of 8 vertices of the large cube. Each line has its other end at another vertex, and so the total number of these lines is $7 \times 8/2 = 28$.

There are 3 lines starting at the midpoint of each of 12 edges of the large cube. Each line has its other end at another midpoint, and so the total number of these lines is $3 \times 12/2 = 18$.

There is one line starting at the midpoint of each of 6 faces of the large cube. Each line has its other end at the midpoint of another face, and so the total number of these lines is $1 \times 6/2 = 3$.

Hence the total number of lines is 49.