2014-2015 Junior Solutions Round 2

J1

In a singles tennis tournament there are 10 players. The organiser needs to arrange the 10 players into 5 pairs for the first round. In how many ways can this first round be drawn up?

Solution

Select one player. Then there are 9 possible choices for his opponent.

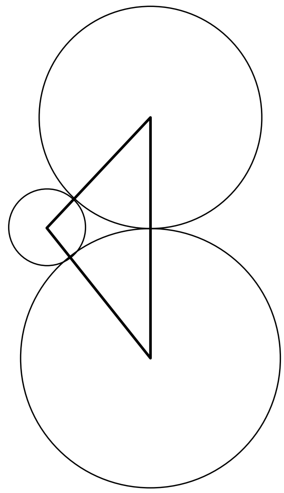
Now select any one of the remaining players. There are 7 choices left for his opponent.

… until the last two players form the last pair.

So the number of ways is $9 \times 7 \times 5 \times 3 \times 1 = 945$.

J2

In the diagram (which is not drawn to scale) the lengths of the sides of the triangle are 8, 9 and 13 centimetres. The centres of the circles are at the vertices of the triangle, and the circles just touch. Find the radius of the largest circle.



Solution

Let the radii of the large, medium and small circles be l, m and s centimetres respectively.

Then

$$l + m = 13$$

$$l + s = 9$$

$$m + s = 8$$

Adding the first two equations:

$$2l + m + s = 22$$

$$2l = 22 - (m + s) = 22 - 8 = 14$$

$$l = 7$$

i.e. the largest circle has radius 7 cm.

Alternative Solution

Let the radii of the large, medium and small circles be l, m and s centimetres respectively.

Then

$$l + m = 13$$

$$l + s = 9$$

$$m + s = 8$$

Adding these gives

$$2l + 2m + 2s = 13 + 9 + 8 = 30$$

$$\text{so} \qquad l + m + s = 15$$

$$\text{using } l + m = 13 \text{ gives } s = 2$$

$$\text{using } l + s = 9 \text{ gives } m = 6$$

$$\text{using } m + s = 8 \text{ gives } l = 7$$

i.e. the largest circle has radius 7 cm.

J3

In a town some of the animals are really strange. Ten percent of the cats think they are dogs, and ten percent of the dogs think they are cats. All the other cats and dogs are perfectly normal. One day I tested all the cats and dogs in the town and found that 20% of them thought that they were cats. What percentage of them really were cats?

Solution

Let the proportion of cats be c and the proportion of dogs be d. Then

$$c + d = 1$$

$$0.9c + 0.1d = 0.2$$

Hence

$$9c + d = 2$$

$$8c = 1$$

$$c = \frac{1}{8} = 12.5\%$$

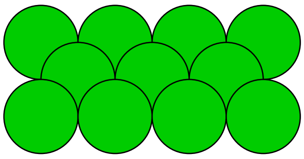
i.e. 12.5% are cats.

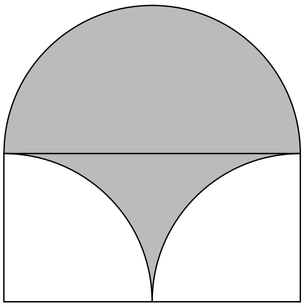
J4

The scales of a large fish are made up of arcs of circles with radius r cm. Each row of scales overlaps the row below. The scales within a row just touch, with their centres on a straight line. The next row of scales, which overlaps the previous row, is r cm above the previous row, with the centres of the scales above the points where the scales in the previous row touch.

What is the visible area of a single scale?

Solution





The area of the semicircle is $\frac{1}{2}\pi r^2$.

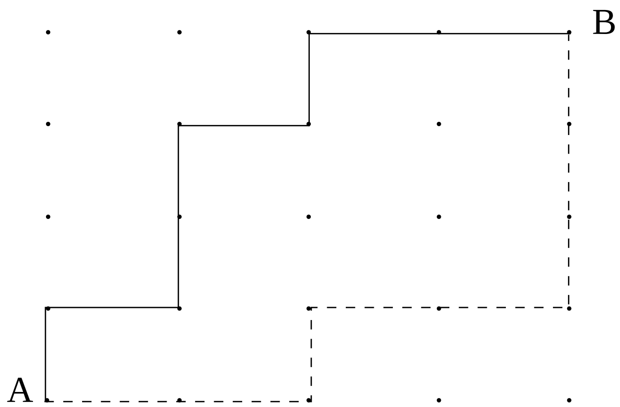
The rectangle is $2 \times 2r$ so its area is $2r^2$.

So the visible area of the top rectangle is $2r^2 - \frac{1}{2}\pi r^2$.

Hence the shaded area is

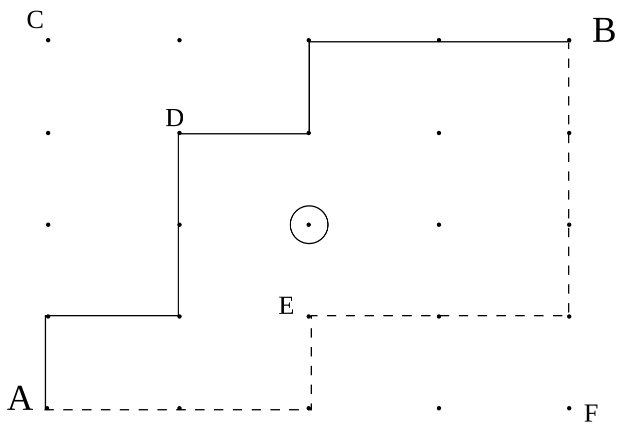
$$\frac{1}{2}\pi r^2 + \left(2r^2 - \frac{1}{2}\pi r^2\right) = 2r^2.$$

J5



Paths from A to B can **only** proceed upwards or to the right: two example paths from A to B are shown. How many such paths are there from A to B that do not go through the centre dot?

Solution



There is only 1 path from A to B via C (or F).

There are 4 paths from A to D (one with its right moving section at each of the 4 levels). Each of these can be combined with one of the 4 paths from D to B, making $4 \times 4 = 16$ possible paths via D (or E).

Thus there are 1 + 16 + 16 + 1 = 34 paths in all not passing through the centre point.