2014-2015 Junior Round 1 solutions

J1

How many whole numbers between 1 and 1000 do not contain the digit 1?

Solution

A reasonable strategy is count the numbers which contain 1.

Between 1 and 99, numbers 1, 10 to 19, and 21, 31, … 91 contain 1 (at least once), i.e. 19 numbers.

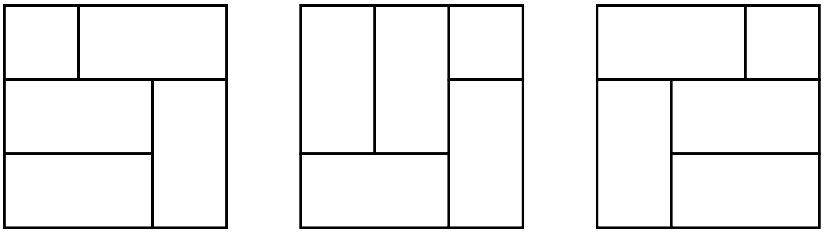
Numbers from 100-199 all contain the digit 1 i.e. 100 in total.

Between 200 and 299 a further 19 numbers contain the digit 1, and similarly for each further hundred, i.e. 8×19 in all.

Finally, 1000 contains the digit 1.

Overall $9 \times 19 + 100 + 1 = 272$ contain the digit 1, and so $1000 - 272 = 728$ do not contain the digit 1.

J2

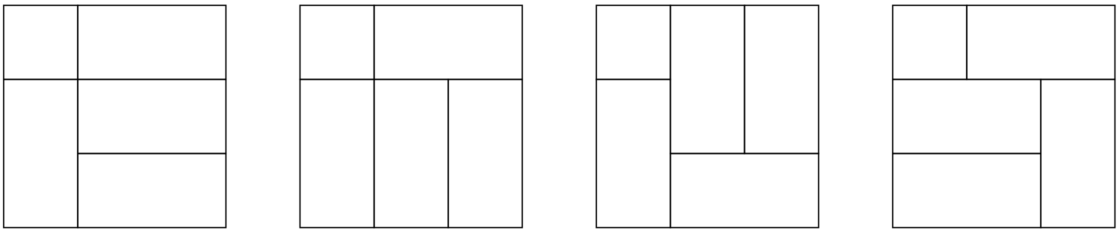


Three different ways of dividing a 3×3 square into one 1×1 square and four 2×1 rectangles are shown above. How many ways are there in all (including the ones shown above)?

Solution

The second diagram is a rotation of the first, and the third is a mirror image of the first. So we must count mirror images and rotations separately.

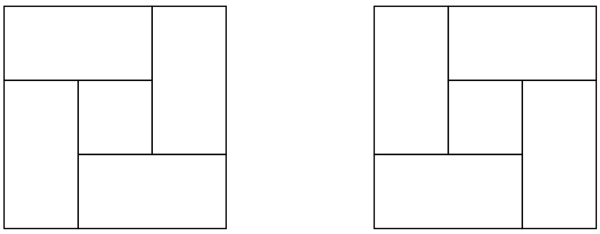
There are 4 ways with the 1×1 square in the top left corner:



and hence $4 \times 4 = 16$ ways with a 1×1 square in any corner.

It is not possible to cover the 3×3 square when the 1×1 square is in the centre of an edge.

There are 2 ways with the 1×1 square in the centre, which are mirror images:



This gives a total of 18 ways.

J3

Simon and Paul set out on their bicycles at the same time and from the same place to ride to a nearby swimming pool. Simon rode three times as long as Paul rested on the trip. Paul rode four times as long as Simon rested. The two reached the pool at the same time. Who rode faster? **Explain your reasoning.**

Solution

Suppose that Simon rode for s hours and Paul rode for p hours.

So Paul rested for $\frac{1}{3}s$ hours and Simon rested for $\frac{1}{4}p$ hours.

Hence $s + \frac{1}{4}p = p + \frac{1}{3}s$ which gives $s = \frac{9}{8}p$.

That is, Simon took longer than Paul, so Paul rode faster.

J4

A shop sells sweets in bags of 7 and 20. What is the largest number of sweets that cannot be purchased exactly? **Justify your answer.**

Solution

7, 14, 21, 28, … can all be purchased in bags of 7 sweets i.e. all amounts that are multiples of 7.

If I start with a bag of 20 sweets, I can purchase 20, 27, 34, … i.e. amounts one less than a multiple of 7.

If I start with 2 bags of 20 sweets, I can purchase 40, 47, 54, … i.e. amounts two less than a multiple of 7.

…

If I start with 6 bags of 20 sweets, I can purchase 120, 127, 134, … i.e. amounts six less than a multiple of 7.

119 is a multiple of 7 so I can purchase all amounts more than 120.

And the largest amount I cannot purchase is 120 - 7 = 113 sweets.

J5

If a year had only 364 days then we could use the same calendar every year. But actually most years have 365 days, and leap years have 366 days. For the relevant years, a leap year occurs when the year is divisible by 4.

I was just about to throw away my calendar for 2014 when I wondered when I would first be able to reuse it. In which year will that be? **Justify your answer.**

Solution

Use the 1st March as the reference day so that the leap day is added in the leap year.

year		1st March day
2014	n	Saturday
2015	$n + 1$	Sunday
2016	$n + 3$	Tuesday
2017	$n + 4$	Wednesday
2018	$n + 5$	Thursday
2019	$n + 6$	Friday
2020	$n + 8 \equiv n + 1$	Sunday
2021	$n + 2$	Monday
2022	$n + 3$	Tuesday
2023	$n + 4$	Wednesday
2024	$n + 6$	Friday
2025	$n + 7 = n$	Saturday

So 2025 has the 1st March on the same day as 2014. Neither year is a leap year, and so the 2014 calendar can be used again in 2025.