2011-12 Junior Set 2 solutions

J1. A water-tank can be filled by any combination of three different taps. With the smallest tap the tank can be filled in 20 minutes. With the middle tap the tank can be filled in 12 minutes. With the largest tap the tank can be filled in 5 minutes. How long does it take to fill the tank with all three taps running?
Explain your reasoning.

Solution
In 1 minute, the smallest tap fills $\frac{1}{20}$ of the tank.
In 1 minute, the middle tap fills $\frac{1}{12}$ of the tank.
In 1 minute, the largest tap fills $\frac{1}{5}$ of the tank.
So in 1 minute, running together, the amount they fill is
$\frac{1}{20} + \frac{1}{12} + \frac{1}{5} = \frac{3}{60} + \frac{5}{60} + \frac{12}{60} = \frac{20}{60} = \frac{1}{3}$ of the tank.
Therefore, the whole water-butt is filled in 3 minutes.

J2. The children were playing with the bouncing balls in the playground when I arrived there. A ball reached my full height before falling back down and I am 1.6 metres tall. “Wow! That was some bounce!”, I said, but the boy responded, “That was nothing − you should have seen the first bounce. That was the third bounce”. Each bounce of this particular ball is 20% less than its previous bounce. What height was the first bounce?
Explain your reasoning.

Solution
20% is $\frac{1}{5}$ so each bounce is $\frac{4}{5}$ of the previous one.
So if the first bounce was x metres, the second one was $\frac{4}{5}x$ metres.
So the the third bounce would have been $\frac{4}{5} \times \frac{4}{5}x = \frac{16}{25}x$ metres.

$$\begin{aligned} \frac{16}{25}x &= 1.6 \\ \frac{1}{25}x &= 0.1 \\ x &= 0.1 \times 25 \\ &= 2.5 \end{aligned}$$

i.e. the first bounce was 2.5 metres.

J3. Five schoolchildren are weighed in pairs. The readings (in kilograms) are:

58, 60, 61, 64, 64, 65, 67, 68, 70 and 71.

Find the weights of the individual children.

Solution

If two children had the same weight then there would be three pairs of repeated weights. Here there is only one repeated weight, and so all the individual weights are different.

Each child is weighed four times in all, so the total of all the pairs is four times the total of the individual weights which is 648 kg. Hence

$$\text{total of individual weights} = \tfrac{1}{4} \times 648 \text{ kg} = 162 \text{ kg}$$

The two lightest children weigh 58 kg together.

The two heaviest children weigh 71 kg together.

So the middle weight child alone weighs $(162 - 58 - 71)$ kg $= 33$ kg.

The second smallest of the paired weights must be of the lightest and middle weight children.

So the lightest child weighs $(60 - 33)$ kg $= 27$ kg.

Then the second lightest child weighs $(58 - 27)$ kg $= 31$ kg.

The second largest of the paired weights must be of the heaviest and middle weight children.

So the heaviest child weighs $(70 - 33)$ kg $= 37$ kg.

Then the second heaviest child weighs $(71 - 37)$ kg $= 34$ kg.

The weights are (in kg): 27; 31; 33; 34; 37.

(This could also be expressed algebraically.)

J4. A set of cards, numbered from 1 to 19, are placed face down on a table. Nine players each pick up two cards. The remaining card is then turned over. The player who achieves the highest total with their two cards plus the number on the remaining card is the winner.

Is it possible for all nine players to have the same total?

If so, what can this total be?

Explain your reasoning.

Solution

The total of the values 1 to 19 is 190. Let n be the centre value. This value will be used a total of nine times (to form the nine possible triples). An expression for the total of all nine triples is the total of the 19 cards plus the duplications of the centre card, i.e. $8n + 190$. Since $8n + 190$ may be written as $9n + 189 - n + 1 = 9(n + 21) + 1 - n$, $1 - n$ has to be divisible by 9. This gives values of n of 1, 10, 19.

n	1	10	19
$8n + 190$	198	270	342
Totals	22	30	38

Each of these leads to possible solutions:

Centre card	Total	Other pairs
1	22	(2,19) (3,18) (4,17) (5,16) (6,15) (7,14) (8,13) (9,12) (10,11)
10	30	(1,19) (2,18) (3,17) (4,16) (5,15) (6,14) (7,13) (8,12) (9,11)
19	38	(1,18) (2,17) (3,16) (4,15) (5,14) (6,13) (7,12) (8,11) (9,10)

Yes, it is possible for all nine players to have the same total and there are three totals 22, 30, 38 as shown.

J5. A computer whizz claims that his program has found some numbers which satisfy Fermat's equation $x^n + y^n = z^n$ for a large integer n.

He tells his 10 year old brother that

$x = 31415926536$ $\quad y = 89173261421$ $\quad z = 90354441655$

Almost immediately his brother says that there cannot be any value of n which will work for these numbers. The computer whizz checks his program and finds a bug.

How did his brother know there was a bug?

Solution

We consider the values of the final digit in the power of each number.

Any power of a number ending in 6 also ends in 6 and so x^n ends in 6.

Any power of a number ending in 1 also ends in 1 and so y^n ends in 1.

So the total of x^n and y^n must end in 6 + 1 = 7.

Any power of a number ending in 5 also ends in 5 and so z^n ends in 5.

Thus the last digits of the two sides of the equation do not match, and so the numbers given must be incorrect.