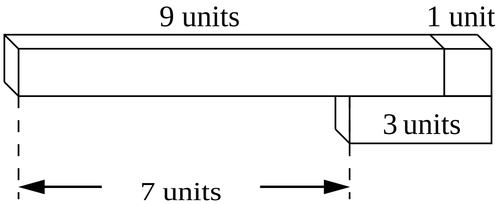
2010 Junior Set 2 solutions

J1. You are given three rods of lengths 1, 3 and 9 units. Using these rods, you could measure 7 units as shown.

Show how you could measure each whole number length up to 13 units.



By adding a fourth rod, it is possible to measure all whole number lengths up to 40 units. What is the length of this extra rod?

Explain your answer.

Solution

Length	1	2	3	4	5	6	7	8
Rods	1	3 − 1	3	3 + 1	9 − 3 − 1	9 − 3	9 − 3 + 1	9 − 1
	9	10	11	12	13			
	9	9 + 1	9 + 3 − 1	9 + 3	9 + 3 + 1			

Add a 27 unit rod so that 14 = 27 – 13, 15 = 27 – 12, . . . , 39 = 27 + 12, 40 = 27 + 13

J2.

You have three boxes, each containing two *identically* wrapped Easter eggs.

One box contains two milk chocolate eggs (M), one contains two plain chocolate eggs (P) and the third contains one *milk chocolate* egg and one *plain chocolate* egg. The boxes are labelled MM, PP or MP according to their contents.

However, someone has switched all the labels so that every box is now incorrectly labelled.

You are allowed to take out one egg at a time *from* any box, check what type it is and put it back. By doing this you can correctly label all three boxes.

What is the smallest number of eggs you would need to check in order to label the boxes correctly?

Explain your answer.

Solution

Labelled boxes: MM, MP and PP all incorrect.

Labels	MM	MP	PP
Actual contents	MP or PP	MM or PP	MP or MM

If you pick an egg from the MM labelled box and it is P then you don't know if it contains MP or PP.

If you pick an egg from the PP labelled box and it is M then you don't know if it contains MP or MM.

However, if you pick an egg from the MP labelled box then

If it is P you know that the box contains PP – solution below

Labels	MP	MM	PP
Selection	P		
Actual contents	PP	MP	MM

And if you pick out a M then you know that it has to be MM giving the solution below

Labels	MP	PP	MM
Selection	M		
Actual contents	MM	MP	PP

So the smallest number of samples is to take one from the box incorrectly labelled MP.

J3. Take all the prime numbers between 30 and 60 and place them in a row in such a way that:

(a) the sum of the two largest numbers and the numbers between them in the row is 233;

(b) the sum of the smallest prime, the middle one in size and the numbers in between them in the row is 133;

(c) the difference between the first and last primes in the row is 6;

(d) the difference between the second and sixth primes in the row is also 6.

Justify your conclusion.

Solution

The seven primes between 30 and 60 are 31, 37, 41, 43, 47, 53, 59.

(a) $233 - (53 + 59) = 121 = 43 + (37 + 41) = 43 + (31 + 47)$.

So 43 lies between 59 and 53 and either the pair (37, 41) or the pair (31, 47) also lie between 59 and 53.

(b) Smallest prime is 31 and the middle one is 43.

$133 - (31 + 43) = 59$.

Now 59 is not the sum of two smaller primes on the list. So 59 must lie between 31 and 43. But 43 is between 59 and 53 so 31 is not between 59 and 53. So the primes between 59 and 53 are 43, 37, 41.

(c) The following pairs of primes differ by 6: (37, 31), (43, 37), (47, 41) , (53, 47), (59, 53).

But since 43, 37, 41 all lie between 59 and 53 they cannot be at either end. This leaves the pairs (53, 47) and (59, 53). But it cannot be (59, 53) as we saw at (a) that not all the numbers lie between them.. So the end pair must be (53,47). Recall that 31,47 lie outside the five numbers (inclusive) between 59 and 53. So 31 and 47 cannot lie at opposite ends. Thus we have deduced that the order will be $47 - 31 - 59 - 43 - x - y - 53$, where $(x, y) = (41, 37)$ or $(x, y) = (37, 41)$.

(The order could, of course, be completely reversed)

(d) This tells us that the one next to the right-hand end is 37 so we finish up with the order

$$47 - 31 - 59 - 43 - 41 - 37 - 53.$$

J4. Four cards, each numbered with a different whole number, are placed face down. Four people, Gavin, Jack, Katie and Luke, in turn select two of these cards, write down their total, and then replace the two cards. Gavin's total is 6, Jack's 9, Katie's 12 and Luke's 15.

Two of the cards are then turned over and their total is 11.

Determine the numbers on each of the cards.

Solution

Let the values of the four cards a, b, c, and d.

There are 6 possible combinations of two cards which can be selected

$$\begin{array}{cc} a,\ b & c,\ d \\ a,\ c & b,\ d \\ a,\ d & b,\ c \end{array} \qquad (1)$$

These can be combined to give three combinations of a, b, c, d, all of which have the same total.

$$a,\ b,\ c,\ d \qquad a,\ c,\ b,\ d \qquad a,\ d,\ b,\ c$$

Now we know that 5 of the six pairs in (1) give totals of 6, 9, 12, 15, 11. Taking combinations of these pairs

6 + 9 = 15	9 + 12 = 21	12 + 15 = 27	15 + 11 = 26
6 + 12 = 18	9 + 15 = 24	12 + 11 = 23	
6 + 15 = 21	9 + 11 = 20		
6 + 11 = 17			

As 21 is the only total which appears twice then $a + b + c + d = 21$ and the missing total is $21 - 11 = 10$.

Suppose that a, b, c, d are listed in order then

$$\begin{array}{rclcrcl} a + b &=& 6 & \qquad & c + d &=& 15 \\ a + c &=& 9 & & b + d &=& 12 \\ a &=& 6 - b & & d &=& 12 - b \end{array}$$

So,

$$(6 - b) + c = 9 \qquad c + (12 - b) = 15.$$

Hence $c - b = 3$ and either

$$b + c = 10 \text{ (not possible)} \qquad \text{or} \qquad b + c = 11 \text{ (so } c = 7 \text{ and } b = 4).$$

This leads to $a = 2$ and $d = 8$.

The numbers on the cards are: 2, 4, 7, 8.

J5. An old-fashioned rectangular billiard table has only four pockets, one at each corner. The lengths of the sides of the table form a whole number ratio.

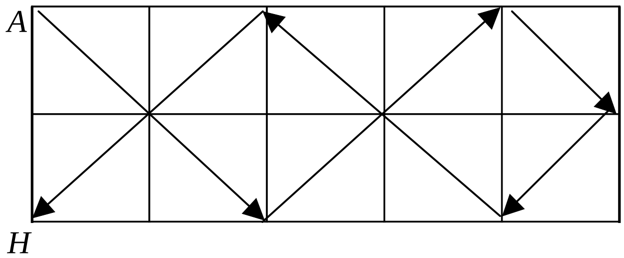
Show that, if the ratio is 5 : 2 and a ball is hit from one corner at an angle of 45°, it will land in a pocket after 5 rebounds.

If the ratio of the sides were $m : n$, where m and n are different whole numbers, with no common factor, and the ball were hit from a corner at an angle of 45°, show that the ball would always drop into a pocket after a number of rebounds. How many rebounds would there be in this case?

Solution

For a 5×2 table, starting at A, we have

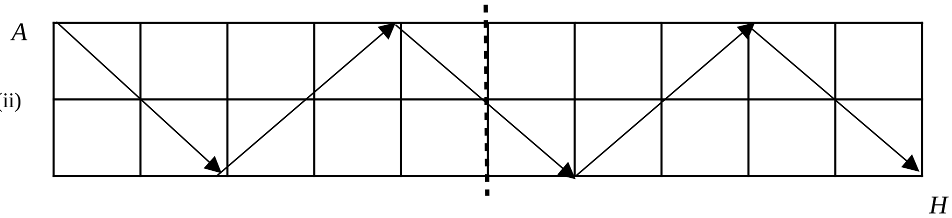
fig (i)



number of rebounds = 5.

Note that the movement could be conveniently represented on a 10×2 grid, as below, where the right half, when folded back over the left half, shows the movement from right to left.

fig (ii)



For any $m \times n$ table, using a diagram as in fig (ii) with a $2m \times n$ grid, there would be $n - 1$ sections lying over the first and there would be an impact on each, giving $n - 1$ impacts on the ends of the table; similarly, from, fig (ii), there will be m movements of the ball to take it across the grid, all except the last ending in a bounce so there are $m - 1$ rebounds on the long edges. Hence the total number of rebounds is $m + n - 2$.