2008-2009 Junior Division Set 2 solutions

J1. Two ferry terminals are directly opposite each other on the Hudson River. At the same instant, a ferry leaves each terminal to cross to the other side. One boat is faster than the other and they meet at a point 650 metres from one bank. After arriving at their destinations, each boat remains for 10 minutes to change passengers and then sets out on the return journey. This time they meet at a point 350 metres from the other bank. How wide is the river?

Solution 1

Since each boat spends the same time at rest, we can ignore that time and assume that they simply turn round and sail back immediately. Let one boat travel at M_1 metres/minute and the other at M_2 metres/minute. Let the width of the river be W metres. Let the boats first meet after t_1 minutes. So $M_1 t_1 = W - 650$, $M_2 t_1 = 650$. Thus $(M_1 + M_2) t_1 = W$. Let the boats meet for the second time after t_2 minutes. So $M_1 t_2 = 2W - 350$, $M_2 t_2 = 350 + W$. Thus $(M_1 + M_2) t_2 = 3W$. So $t_2 = 3t_1$. Thus $M_2 t_2 = 3 \times M_2 t_1 = 3 \times 650 = 1950 = W + 350$. So $W = 1600$.

Solution 2 (minimal algebra)

Draw a diagram of the river. So *total* distance the two ferrys covered when they first met is W, the width of the river. Total distance covered when they met for the second time is $3W$. So one boat goes 3 times as far to the second meeting as to the first. But the distance to first meeting is 650 and to the second $W + 350$.

So $3 \times 650 = W + 350$. So $W = 1600$.

J2. A contractor was planning a small extension to a house and was subcontracting out the work. For some peculiar reason, he worked out what he would have to pay in total to various pairings of subcontractors as follows:

a) £1,000 to the wall-paperer and the painter,

b) £1,700 to the painter and the plumber,

c) £1,100 to the plumber and the electrician,

d) £3,300 to the electrician and the joiner,

e) £5,300 to the joiner and the mason,

f) £3,200 to the mason and the painter.

How much did he pay to each tradesman?

Solution

One of several possible methods of solution is:

From b) and c), the painter gets £600 more that the electrician.

From d) and e), the mason gets £2,000 more than the electrician.

So the mason gets £1,400 more than the painter.

From f), the mason and the painter together get £3,200.

So the painter gets £900 and the mason gets £2,300.

From e), the joiner gets £3,000.

From d), the electrician gets £300.

From c), the plumber gets £800.

From a), the wall-paperer gets £100.

J3. A man who has stolen a horse, rode away on its back. He had gone 6 miles when the owner discovered the theft and started to pursue the thief on his other horse. He chased the thief for 20 miles before he gave up believing that he would be unable to catch the thief.

But it turned out that when he gave up, the thief was only 2 miles in front of him. If he had continued to chase the thief, how many more miles would he have had to ride to catch up with the thief? (The thief and the owner ride at constant speeds).

Solution

Let T be the number of hours the owner rides until he gives up. So his speed is $20/T$.

The distance the thief has covered when the owner gives up is $(20 - 6 + 2)$ miles, i.e. 16 miles. So the speed of the thief is $16/T$.

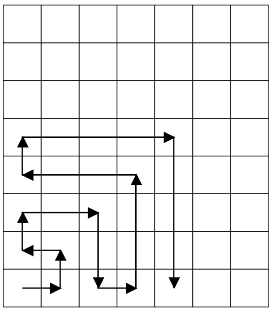
Suppose that it would take the owner a total of h hours to catch the thief. Then

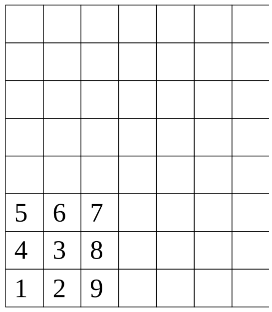
$$\frac{20}{T}h = \frac{16}{T}h + 6$$

$$\text{i.e. } 20\frac{h}{T} = 16\frac{h}{T} + 6 \Rightarrow 4\frac{h}{T} = 6.$$

This gives $\frac{h}{T} = \frac{3}{2}$. So total distance which the owner would need to cover is $\frac{20}{T}h = 30$ miles. So he would have caught the thief in a further 10 miles.

J4. Shaun starts to write down the natural numbers in the square cells of a very large piece of graph paper. He starts at the bottom left corner and writes down the numbers using the following arrangement.





We will identify each of the cells using co-ordinates (x, y) where x is the number of positions to the right and y is the number of position up from the bottom. For example, the cell containing the number 8 has the co-ordinates (3,2).

If N is an even number, what number appears in the cell with co-ordinates $(1, N)$? In what cell does the number 2009 appear? Explain your answers.

Solution

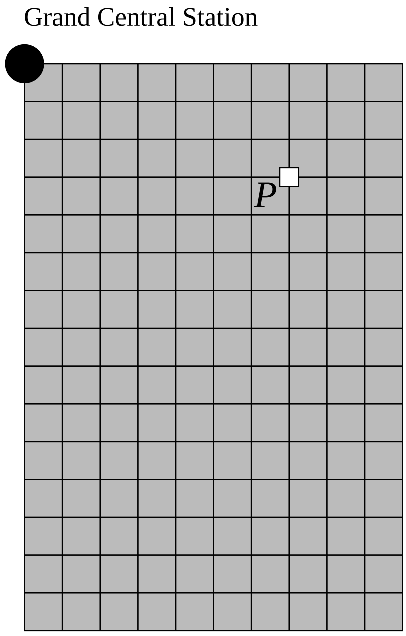
When the path reaches the cell $(1, N)$ for N an even number, it has passed through all cells in the bottom left $N \times N$ block of the graph paper. So the number appearing there will be N^2.

From cell $(1, N)$ the path traverses the cells $(1, N+1), (2, N+1), \ldots, (N+1, N+1)$ along a horizontal row and then down a vertical column traversing the cells $(N+1, N), (N+1, N-1), \ldots, ((N+1,1)$.

Now $2009 = 44^2 + 73 = 44^2 + 45 + 28$ so that 2009 will lie in cell $(45, 45 - 28) = (45, 17)$.

J5. The street system in New York is built up as a series of blocks. The section in which Gordon works is 10 blocks wide and 15 blocks long and Grand Central Station is located in the top north-west corner of the section. When asked where exactly he worked, he would not specify the location, but said that from Grand Central station, starting on January 1st 2009, he could take a different route to work every day except Christmas Day (which he took off anyway!) but that on the January 1st 2010, he would need to repeat a route already used. If Gordon only walks either south or east, find out where he works.

Give your answer as grid location from the station, for example, P is 3 blocks south, 7 blocks east.



Solution

In 2009 there are 365 days but Gordon does not go to work on Christmas Day. So there must be exactly 364 routes from the station to where Gordon works.

The table below shows the number of routes from the station to each point of intersection. (There is just one route to each intersection which is on an edge, thereafter, the number of routes is the sum of the number above and the number to the left - similar to Pascal's triangle.)

blocks	*east*	1	2	3	4	5	6	7	8	9	10
south	station	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	506
5	1	6	21	56	126	252	462	792			
6	1	7	28	84	210	462					
7	1	8	36	120	330						
8	1	9	45	165	495						
9	1	10	55	220							
10	1	11	66	286							
11	1	12	78	**364**							
12	1	13	91								
13	1	14	105								
14	1	15									

It is necessary, in each row and each column to reach 364 (or to exceed it).

This shows that Gordon has to get to 11 blocks south, 3 blocks east.