2007 Junior Set 2 solutions

J1. Oor Wullie and his pals are exploring in the jungle and have to cross a rope bridge at midnight. Unfortunately the bridge is only strong enough to support two people at a time. As it is dark, they also need a torch to be used every time the bridge is crossed but they only have one torch. Wullie can cross the bridge in five minutes, Wee Eck can cross in seven minutes and Fat Bob can cross in eleven minutes. But it takes PC Murdoch twenty minutes to get across. How quickly can all four get across the bridge?

Solution

We need to note that whenever two cross together, the time taken is that of the slower one. There are two equal, shortest times:

(a)	Wullie and Wee Eck cross together	7 minutes
	Wullie returns with the torch	5 minutes
	Fat Bob and PC Murdoch cross together	20 minutes
	Wee Eck returns with the torch	7 minutes
	Wullie and Wee Eck cross back	7 minutes
		46 minutes

Or, interchanging Wullie and Wee Eck returning with the torch.

(b)	Wullie and Wee Eck cross together	7 minutes
	Wee Eck returns with the torch	7 minutes
	Fat Bob and PC Murdoch cross together	20 minutes
	Wullie returns with the torch	5 minutes
	Wullie and Wee Eck cross back	7 minutes
		46 minutes

If Fat Bob and PC Murdoch do not cross together, for a shortest trip, Wullie should accompany them. Then Wee Eck must also cross with Wullie and that would be a total of 11 plus 20 plus 7 plus twice 5 making 48.

J2. A palindromic number is a number which reads the same backwards and forwards, for example 838 and 14541. As generally we do not write numbers with an initial zero, numbers such as 070 will not be included here.

(a) Which are there more of: 10-digit or 11-digit palindromic numbers?

(b) Which are there more of: 11-digit or 12-digit palindromic numbers?

Explain your reasoning.

Solution

(a) If you start with an 10-digit palindromic number, you can construct an 11-digit palindromic number by putting any digit between the 5th and 6th digits. It is possible to use any of the 10 digits 0, 1, . . , 8, 9 so **there are 10 times as many 11-digit than 10-digit palindromes**.

(b) All 12-digit palindromic numbers have a double digit in the middle and you can only get an 11-digit palindromic number by removing one of these so there are the same number of 11 and 12-digit palindromes.

OR

To construct a 12-digit palindromic number from an 11-digit palindromic number, you need to put a copy of the middle digit immediately after the middle digit. This will give a 12-digit palindromic number but ensures that you have the same number of each.

Another Solution

For a 10-digit palindromic number the first digit must be 1 to 9 and the next four can be 0 to 9.

The rest are then automatically determined. So the number of 10-digit palindromes is $9 \times 10 \times 10 \times 10 \times 10 = 9 \times 10^4$.

For 11 and 12-digit palindromic numbers the first digit must be 1 to 9 and the next five can be 0 to 9. The rest are then automatically determined. So the number of 11-digit palindromes is $9 \times 10 \times 10 \times 10 \times 10 \times 10 = 9 \times 10^5$.

Hence there are more 11-digit palindromic numbers than 10-digit palindromic numbers but equal numbers of 11 and 12-digit palindromic numbers.

J3. In school, Jim is about to take the last of a series of tests. If he gets 23 in this test, it will improve his average by 1. If he scores 39 it will improve his average by 3. What was his average score before the last test?

Solution

23 additional marks improve his average score by 1. A further additional 16 marks would improve his average score by 2 more. From this we see that each extra 8 marks changes his average by 1 so going down by 8 from 23 gives his previous average as 15.

OR

Using algebra: let n represent the number of tests taken (excluding the last one) and x be his total marks at that stage. So

Currrent average = $\frac{x}{n}$

If he gets 23, average = $\frac{x+23}{n+1} = \frac{x}{n} + 1$.

If he gets 39, average = $\frac{x+39}{n+1} = \frac{x}{n} + 3$.

Subtracting:

$$\frac{x+39}{n+1} - \frac{x+23}{n+1} = 2$$

$$\frac{x+39-x-23}{n+1} = 2$$

$$16 = 2n + 2$$

$$n = 7 \text{ and } \frac{x}{8} + \frac{23}{8} = \frac{x}{7} + 1 \Rightarrow 7x + 7 \times 23 = 8x + 56 \Rightarrow x = 161 - 56 = 105$$

so his average prior to his last test is 15.

J4. In the village of Piffle, some of the animals are really strange. Ten percent of the dogs think they are cats and ten percent of the cats think they are dogs. All the other cats and dogs are perfectly normal. One day the official village animal psychologist tested all the cats and dogs in the village and found that twenty percent of them thought that they were cats. What percentage of them really were cats?

Solution

Suppose that there are n cats in the village and m dogs. Then $0.9n$ cats think they are cats and the other $0.1n$ cats think they are dogs. Also $0.9m$ dogs think they are dogs and $0.1m$ dogs think they are cats. So the total number of animals in the village that think they are cats is $0.9n + 0.1m$. Thus the fraction that think they are cats is

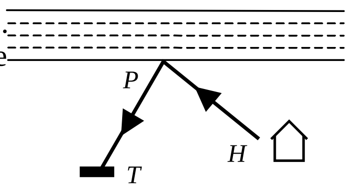
$$\frac{0.9n + 0.1m}{n + m} = 0.2$$

$$\Rightarrow 0.9n + 0.1m = 0.2n + 0.2m$$

$$\Rightarrow 0.7n = 0.1m \Rightarrow m = 7n.$$

But then actual fraction of cats is $\dfrac{n}{n + m} = \dfrac{n}{n + 7n} = \dfrac{1}{8}$. Thus the percentage is 12.5.

J5. Every morning, Ruth goes out to take water to her pet armadillo. She picks up her water bucket by her house at the point H on the diagram below, goes down to the edge of the straight river, fills the bucket with water and takes it to the armadillo's trough at T. She has been doing this so often that she knows exactly where the point P on the river bank is, so that she walks the shortest distance. Explain how you should choose the point P on the river bank so that the distance from H to T via P is shortest.



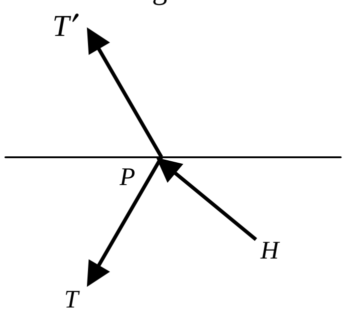
Solution

Regard the edge of the riverbank as a mirror and let T' be the mirror image of T. Then the distance from H to T via P is the same as the distance from H to T' via P.

But the shortest distance between any two points is a straight line (on a flat plane).

So the line from H through P to T' should be a straight line.

So the angle that HP makes with the riverbank will be the same as the angle $T'P$ makes with the riverbank, which, in turn, is the same as the angle TP makes with the riverbank.



The point on the riverbank should be chosen so that the angle that the line HP makes with the bank is the same as the angle that the line TP makes with the riverbank.