2007 Junior Set 1 solutions

J1. Three friends visit a museum and walk up a flight of stairs. Ross goes up one step at a time starting with his left foot on the first step. Sheila goes up two steps at a time starting with her left foot on the second step and Tom starts with his left foot on the third step and goes up three steps at a time.

Investigate these questions and explain your answers.

(a) Which is the first step that all three will tread on?

(b) Which is the first step that all three will tread on with their right foot?

(c) Which is the first step that all three will tread on with their left foot?

Solution

Ross puts his left foot on steps: 1, 3, 5 . . . and his right foot on steps: 2, 4, 6, . . .

Sheila puts her left foot on steps: 2, 6, 10 . . . and her right foot on steps: 4, 8, 12, . . .

Tom puts his left foot on steps: 3, 9, 15, . . . and his right foot on steps: 6, 12, 18, . . .

(a) 6

(b) 12

(c) No solution as Ross steps on odd numbered steps with his left foot and Sheila only steps on even numbered steps with her left foot.

Step	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Ross	L	R	L	R	L	R	L	R	L	R	L	R
Sheila		L		R		L		R		L		R
Tom			L			R			L			R

J2. A group of boys found a chestnut tree with the chestnuts just ready for picking. One of the boys climbed the tree and was able to knock down some chestnuts. He had just enough to give himself and the other boys three chestnuts each, with none left over. Then three of their friends joined them. They found that it was not possible to share the chestnuts evenly among the group.

However, when one more chestnut was picked, it was possible to give each boy two chestnuts, with none left over. How many boys were there altogether? **Explain your reasoning.**

Solution

(a)	Number of boys at start	2	3	4	5	6	. . .
(b)	Number of chestnuts at start	6	9	12	15	18	
(c)	Number of boys with extras	5	6	7	8	9	
(d)	Twice this number of boys	10	12	14	16	18	
	Row (d) − Row (b)	4	3	2	1	0	

The table shows the condition is satisfied when there are 5 boys at the start and 8 boys altogether.

It also shows that as the number of boys increases the final row continually decreases.

Using algebra:

Suppose that there were n boys at the start so that there were $3n$ chestnuts. After the other boys arrived, there were $n + 3$ boys and $3n + 1$ chestnuts.

Hence, $3n + 1 = 2(n + 3)$ so $n = 5$.

Total number of boys = 8.

J3. Show that there is no five-digit number which uses each of the digits 1,2,3,4,5 such that the numbers formed

by the first digit is divisible by 1,
by the first two digits is divisible by 2,
by the first three digits is divisible by 3,
by the first four digits is divisible by 4,
by the first five digits is divisible by 5.

Explain your reasoning.

Solution

Suppose there is such a number. Denote it by *abcde*.

Since it is to be divisible by 5, we must have $e = 5$.

Since *ab* is divisible by 2, *b* must be an even number and in the same way, since *abcd* is divisible by 4, *d* must also be an even number.

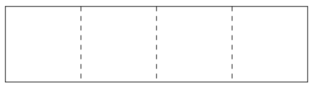
Thus we can now list the possibilities: 12345, 32145, 14325, 34125.

Now 123 and 321 are divisible by 3 but 143 and 341 are not.

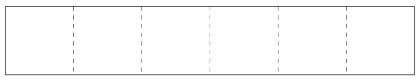
So we are left with 12345 and 32145. But since neither 1234 nor 3214 are divisible by 4, there is no number which satisfies these conditions.

J4. In this question, you are only allowed to shade complete squares.

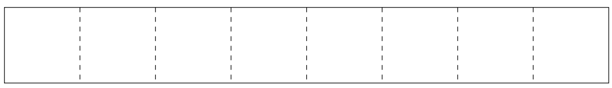
(a) In how many different ways is it possible to shade one half of this rectangle?



(b) In how many different ways is it possible to shade one third of this rectangle?



(c) In how many different ways is it possible to shade one quarter of this rectangle?



(d) In how many different ways is it possible to shade one fortieth of a rectangle made up of 80 squares?

Solution

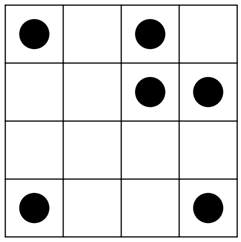
(a) To shade half the rectangle, we need to shade 2 of the four squares. We can choose the two squares out of four in 6 ways.

(b) Again we need to shade 2 squares and there are 15 ways to choose two squares out of 6.

(c) Once again, we need to shade 2 squares and there are 28 ways to choose two squares out of 8.

(d) $\frac{1}{40}$ is still 2 squares. The first square can be any of 80 and the second any of 79. Multiplying 80 and 79 gives 6320 but this counts each pair twice. So there are 3160 ways.

J5. The diagram shows a 4 × 4 grid containing 6 black spots. These 6 spots are so placed that no three of them lie in a line, either horizontally, vertically or diagonally, but if you add one more spot, there will always be such a line of three spots.



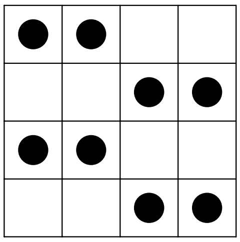
What is the largest number of black spots you can place on such a 4 × 4 grid with this property i.e. no three spots are in a line but if you add any one spot there will always be a line of three?

What is the smallest number of black spots you can place on such a 4 × 4 grid with this property?

Explain your answers.

Solution

The largest number is 8 and these can be placed as shown below, but there are several other possibilities. If you have 9 spots or more there must be at least 3 in one row, so 8 is the maximum.



The smallest number is 4 with the spots placed either at the four corners or in the middle four squares.

If 3 spots were enough, two spots would have to lie in one row and the third in a different row. Now consider a row with no spots in it. The only squares in that row which will give 3 in a line are those which lie on one of the two lines from the two spots in one row to the one spot in the other row. So a black spot could be placed in either of the two remaining squares in that row and that spot would not lie in a line of 3.

OR

If 3 spots were enough then two would have to lie in one row with the third in a different row. But the same applies to columns. So the three would have to lie in squares which form three of the vertices of a rectangle of squares.